

ON RESTRICTED FAMILIES OF PROJECTIONS IN \mathbb{R}^3

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ABSTRACT. We study projections onto non-degenerate one-dimensional families of lines and planes in \mathbb{R}^3 . Using the classical potential theoretic approach of R. Kaufman, one can show that the Hausdorff dimension of at most $1/2$ -dimensional sets $B \subset \mathbb{R}^3$ is typically preserved under one-dimensional families of projections onto lines. We improve the result by an ε , proving that if $\dim_{\text{H}} B = s > 1/2$, then the packing dimension of the projections is almost surely at least $\sigma(s) > 1/2$. For projections onto planes, we obtain a similar bound, with the threshold $1/2$ replaced by 1 . In the special case of self-similar sets $K \subset \mathbb{R}^3$ without rotations, we obtain a full Marstrand type projection theorem for one-parameter families of projections onto lines. The $\dim_{\text{H}} K \leq 1$ case of the result follows from recent work of M. Hochman, but the $\dim_{\text{H}} K > 1$ part is new: with this assumption, we prove that the projections have positive length almost surely.

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1. INTRODUCTION

We start the introduction by defining what we mean by *non-degenerate one-dimensional families* of lines and planes.

Definition 1.1 (Non-degenerate families). Let $U \subset \mathbb{R}$ be an open interval, and let $\gamma : U \rightarrow S^2$ be a \mathcal{C}^3 -curve on the unit sphere in \mathbb{R}^3 satisfying the condition

$$\text{span}(\{\gamma(\theta), \dot{\gamma}(\theta), \ddot{\gamma}(\theta)\}) = \mathbb{R}^3, \quad \theta \in U. \quad (1.2)$$

To each point $\gamma(\theta)$, $\theta \in U$, we assign the line $\ell_\theta = \text{span}(\gamma(\theta))$. Any family of lines $(\ell_\theta)_{\theta \in U}$ so obtained is called a *non-degenerate family of lines*. The orthogonal complements $V_\theta := \ell_\theta^\perp$ form a one-dimensional family of planes. Any family of planes $(V_\theta)_{\theta \in U}$ so obtained is called a *non-degenerate family of planes*.

In this paper, we are concerned with projecting analytic sets $B \subset \mathbb{R}^3$ orthogonally onto non-degenerate families of lines and planes.

Definition 1.3 (Projections ρ_θ and π_θ). If $(\ell_\theta)_{\theta \in U}$ is a non-degenerate family of lines associated with the curve $\gamma : U \rightarrow S^2$, we write $\rho_\theta : \mathbb{R}^3 \rightarrow \mathbb{R}$ for the orthogonal projection

$$\rho_\theta(x) = \gamma(\theta) \cdot x.$$

Thus, we interpret the projection onto the line ℓ_θ spanned by $\gamma(\theta)$ as a subset of \mathbb{R} . Given a non-degenerate family of planes $(V_\theta)_{\theta \in U}$, we denote by $\pi_\theta : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ the orthogonal projection onto the plane V_θ , again interpreted as a subset of \mathbb{R}^2 . Projection families of the form $(\rho_\theta)_{\theta \in U}$ and $(\pi_\theta)_{\theta \in U}$ will be referred to as *non-degenerate families of projections*.

There is some recent literature concerning ‘non-degenerate families of projections’, see [6] and [5]. However, the condition of ‘non-degeneracy’ in these papers is weaker than the one imposed by Definition 1.1, and also the results are rather different in nature. For instance, a family of lines $(\ell_\theta)_{\theta \in U}$ determined by a differentiable curve $\gamma : U \rightarrow S^2$ is ‘non-degenerate’ according to the definition in [5], as soon as $\dot{\gamma}$ is non-vanishing everywhere. So, this definition allows for the ‘non-degenerate family of lines’ to lie on the xy -plane, and then the z -axis is projected to the origin by every projection. Definition 1.1 excludes such counterexamples: in fact, it is instructive to interpret (1.2) as a curvature condition for the surface

$$S = \bigcup_{\theta \in U} \ell_\theta,$$

which makes it impossible for large parts of S to lie on a single plane.

1.1. Dimension estimates for general sets. Unless otherwise stated, $(\rho_\theta)_{\theta \in U}$ and $(\pi_\theta)_{\theta \in U}$ will always stand for non-degenerate families of projections onto lines and planes, respectively. When $\dim_{\mathbb{H}} B$ lies on certain intervals, dimension conservation for non-degenerate families of projections can be proven directly using the classical ‘potential theoretic’ method pioneered by R. Kaufman in [7]. These bounds are the content of the following proposition.

Proposition 1.4. *Let $B \subset \mathbb{R}^3$ be an analytic set.*

- (a) *If $\dim_{\text{H}} B \leq 1/2$, then $\dim_{\text{H}} \rho_{\theta}(B) = \dim_{\text{H}} B$ almost surely.*
- (b) *If $\dim_{\text{H}} B \leq 1$, then $\dim_{\text{H}} \pi_{\theta}(B) = \dim_{\text{H}} B$ almost surely.*

Part (b) follows from a result of Järvenpää, Järvenpää, Ledrappier and Leikas, see [6, Proposition 3.2]. The proof of part (a) is standard: we include it mainly to identify the ‘enemy’ against which we have to combat in order to obtain an improvement, but also because the proof contains certain sub-level estimates needed to prove Theorem 1.6. Before stating any further results, let us make explicit the standing conjecture:

Conjecture 1.5. *In Proposition 1.4(a), the hypothesis $\dim_{\text{H}} B \leq 1/2$ can be relaxed to $\dim_{\text{H}} B \leq 1$. In part (b), the hypothesis $\dim_{\text{H}} B \leq 1$ can be relaxed to $\dim_{\text{H}} B \leq 2$.*

We fall short of proving the conjecture in two ways: first, we are only able to obtain a non-trivial lower bound for the *packing dimension* \dim_{p} (see [11, §5.9]) of the projections, and, second, our bound is much weaker than the full dimension conservation conjectured above.

Theorem 1.6. *Let $B \subset \mathbb{R}^3$ be an analytic set, and write $s := \dim_{\text{H}} B$.*

- (a) *If $s > 1/2$, there exists a number $\sigma_1 = \sigma_1(s) > 1/2$ such that $\dim_{\text{p}} \rho_{\theta}(B) \geq \sigma_1$ almost surely.*
- (b) *If $s > 1$, there exists a number $\sigma_2 = \sigma_2(s) > 1$ such that $\dim_{\text{p}} \pi_{\theta}(B) \geq \sigma_2$ almost surely.*

Remark 1.7. The lower bounds for $\sigma_1(s)$ and $\sigma_2(s)$ given by the proof of Theorem 1.6 are most likely not optimal and certainly not very informative. They are

$$\sigma_1(s) \geq \frac{1}{2} + \frac{1}{2} \cdot \frac{(2s-1)^2}{12s^2 + 4s - 1} \quad \text{and} \quad \sigma_2(s) \geq 1 + \frac{(s-1)^2}{2s-1}.$$

For $\sigma_1(s)$ we also have the easy bound $\sigma_1(s) \geq s/2$, see Proposition 5.4. The two bounds are equal when $s \approx 1.077$.

The proof of Theorem 1.6 involves analysing the (hypothetical) situation where the packing dimension of the projections of B drops in positively many directions very close to the ‘classical’ bounds given by Proposition 1.4. Building on this counter assumption, we extract a large subset of B with additional structure. This information is used to show that the projections of the subset must have fairly large dimension.

1.2. A Marstrand type theorem for self-similar sets. Self-similar sets $K \subset \mathbb{R}^3$ without rotations enjoy the following structural property. If $\pi: \mathbb{R}^3 \rightarrow V$ is the orthogonal projection onto any plane $V \subset \mathbb{R}^3$, there exists a compact subset $\tilde{K} \subset K$ with $\dim_{\text{H}} \tilde{K} \approx \dim_{\text{H}} \pi(K)$ such that the restriction $\pi|_{\tilde{K}}$ is bi-Lipschitz. This property plays a key role in the following improvement of Theorem 1.6.

Theorem 1.8. *Let $K \subset \mathbb{R}^3$ be a self-similar set without rotations.*

- (a) If $0 \leq \dim_{\mathbb{H}} K \leq 1$, then $\dim_{\mathbb{H}} \rho_{\theta}(K) = \dim_{\mathbb{H}} K$ almost surely.
- (b) If $\dim_{\mathbb{H}} K > 1$, then $\rho_{\theta}(K)$ has positive length almost surely.

The phrase ‘self-similar set without rotations’ means that the generating similitudes of K have the form $\psi(x) = rx + w$ for some $r \in (0, 1)$ and $w \in \mathbb{R}^3$. The case of self-similar sets with only ‘rational’ rotations easily reduces to the one with no rotations, but we are not able to prove Theorem 1.8 for arbitrary self-similar sets in \mathbb{R}^3 . Part (a) of Theorem 1.8 follows from recent work of M. Hochman [3], but the methods of proof are very different. Part (b) is new. In addition to the structural property of self-similar sets mentioned above, the proof of part (b) relies on an application of Theorem 1.6(b). Unfortunately, the structural property can completely fail for general sets $B \subset \mathbb{R}^3$, see Remark 4.8, so Theorem 1.8 does not seem to admit further generalisation with our techniques.

Specialising Theorem 1.8 to the family of lines foliating the surface of a vertical cone in \mathbb{R}^3 , one immediately obtains the following corollary:

Corollary 1.9. *Let $K \subset \mathbb{R}$ be an equicontractive self-similar set with $\dim_{\mathbb{H}} K > 1/3$. Then*

$$(\cos \theta) \cdot K + (\sin \theta) \cdot K + K$$

has positive length for almost all $\theta \in [0, 1]$.

A self-similar set is called *equicontractive*, if all the generating similitudes have equal contraction ratios. The proof can be found in Section 5. The corollary is close akin to Theorem 1.1(b), in Y. Peres and B. Solomyak’s paper [15] with the choice $C_{\lambda} = (\cos \lambda) \cdot K + (\sin \lambda) \cdot K$, in the notation of [15]. The self-similar sets C_{λ} treated in [15] are not as specific in form as the ones in Corollary 1.9. On the other hand, the proof in [15] is based on the concept of *transversality* and, hence, requires that the sets C_{λ} satisfy the strong separation condition for all $\lambda \in (0, 1)$. This is generally not the case with the sets $C_{\lambda} = (\cos \lambda) \cdot K + (\sin \lambda) \cdot K$ above.

1.2.1. Notation. Throughout the paper we will write $a \lesssim b$, if $a \leq Cb$ for some constant $C \geq 1$. The two-sided inequality $a \lesssim b \lesssim a$, meaning $a \leq C_1 b \leq C_2 a$, is abbreviated to $a \sim b$. Should we wish to emphasise that the implicit constants depend on a parameter p , we will write $a \lesssim_p b$ and $a \sim_p b$. The closed ball in \mathbb{R}^d with centre x and radius $r > 0$ will be denoted by $B(x, r)$.

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3. GENERAL SETS

In this section, we prove Proposition 1.4 and Theorem 1.6. It suffices to prove all ‘almost sure’ statements for any fixed compact subinterval I of the parameter set U . For the rest of the paper, we assume that this interval is $I = [0, 1]$.

3.1. Classical bounds. The Hausdorff dimension of an analytic set $B \subset \mathbb{R}^n$ with $\dim_{\text{H}} B \leq m$ is preserved under almost every projection onto an m -dimensional subspace in \mathbb{R}^n . This was first proved by J. M. Marstrand [8] in 1954 for $n = 2$ and $m = 1$. Later, in 1968, R. Kaufman [7] found a potential theoretic proof for the same result, using integral averages over energies of projected measures. Kaufman's proof was generalised to arbitrary dimensions by P. Mattila [9] in 1975. It is a natural point of departure for our studies to see if Kaufman's method could be used to prove dimension conservation for non-degenerate families of projections onto lines and planes in \mathbb{R}^3 . For appropriate ranges of $\dim_{\text{H}} B$ – namely when $\dim_{\text{H}} B$ is small enough – the answer is positive. This is the content of Proposition 1.4.

Proof of Proposition 1.4. We discuss first part (a) of the proposition, which concerns projections onto lines. Since orthogonal projections are Lipschitz continuous, the upper bound $\dim_{\text{H}} \rho_{\theta}(B) \leq \dim_{\text{H}} B$ holds for every parameter θ . To establish the almost sure lower bound $\dim_{\text{H}} \rho_{\theta}(B) \geq \dim_{\text{H}} B$, we use the standard potential-theoretic method. Let $t < \dim_{\text{H}} B \leq 1/2$ and find a positive and finite Borel regular measure μ which is supported on B and whose t -energy is finite,

$$I_t(\mu) = \int \int |x - y|^{-t} d\mu(x) d\mu(y) < \infty.$$

Such a measure exists by Frostman's lemma for analytic sets, see [1]. For each $\theta \in [0, 1]$, the push-forward measure $\mu_{\theta} = \rho_{\theta\#} \mu$ defined by $\mu_{\theta}(E) := \mu(\rho_{\theta}^{-1}(E))$ is a measure supported on $\rho_{\theta}(B)$. Our goal is to prove that $\int_0^1 I_t(\mu_{\theta}) d\theta < \infty$, which implies $I_t(\mu_{\theta}) < \infty$ and thus $\dim_{\text{H}} \rho_{\theta}(B) \geq t$ for almost every $\theta \in [0, 1]$. Using Fubini's theorem, we find

$$\begin{aligned} \int_0^1 I_t(\mu_{\theta}) d\theta &= \int_0^1 \int \int |u - v|^{-t} d\mu_{\theta}(u) d\mu_{\theta}(v) \\ &= \int_B \int_B \left(\int_0^1 |\rho_{\theta}(x - y)|^{-t} d\theta \right) d\mu(x) d\mu(y) \\ &\lesssim \int_B \int_B |x - y|^{-t} d\mu(x) d\mu(y) = I_t(\mu). \end{aligned}$$

The inequality follows from the next lemma combined with the linearity of ρ_{θ} .

Lemma 3.1. *Given $t < 1/2$, the estimate*

$$\int_0^1 |\rho_{\theta}(x)|^{-t} d\theta \lesssim_t 1$$

holds for all $x \in S^2$.

Proof of Lemma 3.1. We consider the function

$$\Pi : [0, 1] \times S^2 \rightarrow \mathbb{R}, \quad \Pi(\theta, x) := \rho_{\theta}(x) = x \cdot \gamma(\theta).$$

To prove the lemma, we fix $x \in S^2$ and study the behaviour of $\theta \mapsto \Pi(\theta, x)$. We note that

$$\partial_\theta \Pi(\theta, x_0) = x \cdot \dot{\gamma}(\theta) \quad \text{and} \quad \partial_\theta^2 \Pi(\theta, x) = x \cdot \ddot{\gamma}(\theta).$$

If $\Pi(\theta, x) = \partial_\theta \Pi(\theta, x) = 0$ for some $x \in S^2$ and $\theta \in [0, 1]$, we infer that x is orthogonal to both $\gamma(\theta)$ and $\dot{\gamma}(\theta)$. If this happens, the second derivative $\partial_\theta^2 \Pi(\theta, x)$ cannot vanish, because then x would be orthogonal to $\ddot{\gamma}(\theta)$ as well, and this is ruled out by the non-degeneracy condition (1.2). We have now shown that

$$\Pi(\theta, x) = \partial_\theta \Pi(\theta, x) = 0 \quad \Rightarrow \quad \partial_\theta^2 \Pi(\theta, x) \neq 0$$

for $(\theta, x) \in [0, 1] \times S^2$. A compactness argument then yields a constant $c > 0$ such that

$$\max\{|\Pi(\theta, x)|, |\partial_\theta \Pi(\theta, x)|, |\partial_\theta^2 \Pi(\theta, x)|\} \geq c, \quad (\theta, x) \in [0, 1] \times S^2. \quad (3.2)$$

Since

$$\int_0^1 |\Pi(\theta, x)|^{-t} d\theta = \int_0^\infty \mathcal{L}^1(\{\theta \in [0, 1] : |\Pi(\theta, x)| \leq r^{-\frac{1}{t}}\}) dr,$$

we will need a uniform estimate for the \mathcal{L}^1 measures of the sub-level sets of $\theta \mapsto |\Pi(\theta, x)|$. Such an estimate is provided for instance by [2, Lemma 3.3]: for every $k \in \mathbb{N}$, there is a constant $C_k < \infty$ so that for every interval $I \subset \mathbb{R}$, every $f \in \mathcal{C}^k(I)$ and every $\lambda > 0$,

$$\mathcal{L}^1(\{\theta \in I : |f(\theta)| \leq \lambda\}) \leq C_k \left(\frac{\lambda}{\inf_{\theta \in I} |\partial_\theta^k f(\theta)|} \right)^{\frac{1}{k}}. \quad (3.3)$$

The next lemma, proved in Appendix B, shows that the mapping $\theta \mapsto \Pi(\theta, x)$ can only have finitely many zeroes on $[0, 1]$.

Lemma 3.4. *Let $\gamma : [0, 1] \rightarrow S^2$ be a parameterized curve satisfying the condition (1.2). Then there exists $\varepsilon > 0$ such that for all $x \in S^2$ the function $\theta \rightarrow \rho_\theta(x)$ vanishes in at most two points in every interval of length ε .*

Recall that in each of the finitely many points $\theta_0 \in [0, 1]$, where $\Pi(\theta_0, x) = 0$, either $\partial_\theta \Pi(\theta_0, x) \neq 0$ or $\partial_\theta^2 \Pi(\theta_0, x) \neq 0$. Now, the uniform continuity of Π and its partial derivatives guarantee that there exists an open ball $U(x)$ centred at x with the following property: $[0, 1]$ can be covered by a finite number of intervals $I_1, \dots, I_{n(x)}$, for each of which there are numbers $k_i \in \{0, 1, 2\}$ and $c_i > 0$ with

$$\inf_{\theta \in I_i} |\partial_\theta^{k_i} \Pi(y, \theta)| \geq c_i \quad \text{for all } y \in U(x). \quad (3.5)$$

Set $c_0 := \min\{c_1, \dots, c_{n(x)}\}$. The sub-level set estimate (3.3) applied to this situation yields

$$\mathcal{L}^1(\{\theta \in I_i : |\Pi(\theta, y)| \leq \lambda\}) \leq \begin{cases} 0 & \text{if } k_i = 0, \lambda \leq c_0 \\ C_{k_i} \left(\frac{\lambda}{c_i} \right)^{\frac{1}{k_i}} & \text{if } k_i \in \{1, 2\} \end{cases}$$

for all $i \in \{1, \dots, n(x)\}$ and $y \in U(x)$. Hence, there exists a finite constant $c(x) > 0$ such that

$$\mathcal{L}^1(\{\theta \in [0, 1] : |\Pi(\theta, y)| \leq \lambda\}) \leq c(x)\lambda^{\frac{1}{2}}, \quad \text{for all } y \in U(x) \text{ and } 0 < \lambda \leq 1.$$

The sphere S^2 can be covered by finitely many balls of the form $U(x)$, so

$$\mathcal{L}^1(\{\theta \in [0, 1] : |\Pi(\theta, x)| \leq \lambda\}) \lesssim \lambda^{\frac{1}{2}}, \quad \text{for all } x \in S^2 \text{ and } 0 < \lambda \leq 1. \quad (3.6)$$

Finally, we obtain

$$\int_0^1 |\Pi(\theta, x)|^{-t} d\theta = \int_0^\infty \mathcal{L}^1(\{\theta \in [0, 1] : |\Pi(\theta, x)| \leq r^{-\frac{1}{t}}\}) dr \lesssim 1 + \int_1^\infty r^{-\frac{1}{2t}} dr,$$

where the right hand side is finite by the assumption $t < 1/2$. \square

It remains to prove part (b) of the proposition. The result follows directly from [6, Proposition 3.2] or [5, Theorem 3.2], since the family $(\pi_\theta)_{\theta \in U}$ is ‘full’ or ‘non-degenerate’ in the senses of [6] and [5]. We will need the sub-level estimate (3.9) later, so we choose to include the proof. As in (a), we are reduced to proving the uniform bound

$$\int_0^1 |\pi_\theta(x)|^{-t} d\theta = \int_0^\infty \mathcal{L}^1(\{\theta \in [0, 1] : |\pi_\theta(x)|^2 \leq r^{-\frac{2}{t}}\}) dr \lesssim_t 1, \quad x \in S^2, \quad (3.7)$$

valid for $0 < t < 1$. To see this, we note that the function

$$F : [0, 1] \times S^2 \rightarrow \mathbb{R}, \quad F(\theta, x) := |\pi_\theta(x)|^2 = d(x, \ell_\theta)^2 = 1 - \rho_\theta(x)^2$$

can have at most second order zeros. Indeed, the formulae

$$\begin{aligned} F(\theta, x) &= 1 - (x \cdot \gamma(\theta))^2, \quad \partial_\theta F(\theta, x) = -2(x \cdot \dot{\gamma}(\theta))(x \cdot \dot{\gamma}(\theta)), \\ \partial_\theta^2 F(\theta, x) &= -2(x \cdot \ddot{\gamma}(\theta))^2 - 2(x \cdot \dot{\gamma}(\theta))(x \cdot \ddot{\gamma}(\theta)) \end{aligned} \quad (3.8)$$

reveal that if $F(\theta, x) = 0$ for some $\theta \in [0, 1]$ and $x \in S^2$, then x is parallel to $\gamma(\theta)$. This implies that $x \cdot \dot{\gamma}(\theta) = 0$ and $x \cdot \ddot{\gamma}(\theta) \neq 0$ since

$$\gamma \cdot \gamma = 1 \quad \Rightarrow \quad \gamma \cdot \dot{\gamma} = 0 \quad \Rightarrow \quad \gamma \cdot \ddot{\gamma} \neq 0.$$

This means that $\partial_\theta^2 F(\theta, x) \neq 0$, by (3.8). Now, tracing the proof of Lemma 3.4, we conclude that the number of zeros of the function $\theta \mapsto F(\theta, x)$ is finite for $x \in S^2$. Since $|F(\theta, x)| = |\pi_\theta(x)|^2$, the sub-level estimate (3.3) and the compactness of S^2 yield

$$\mathcal{L}^1(\{\theta \in [0, 1] : |\pi_\theta(x)| \leq \lambda\}) \lesssim \lambda, \quad x \in S^2. \quad (3.9)$$

This proves (3.7) for $0 < t < 1$. \square

3.2. Beyond the classical bounds. It can be read from the proof of Proposition 1.4 above, why the potential theoretic method does not directly extend beyond the dimension ranges $0 \leq \dim_{\mathbb{H}} B \leq 1/2$ (for lines) and $0 \leq \dim_{\mathbb{H}} B \leq 1$ (for planes). If $x, y \in \mathbb{R}^3$ are points such that $(x - y) \perp \gamma(\theta_0)$ and $(x - y) \perp \dot{\gamma}(\theta_0)$ for some $\theta_0 \in (0, 1)$, then both the mapping $\theta \mapsto \rho_\theta(x - y)$ and its first derivative have a zero at $\theta = \theta_0$. This means that

$$\int_0^1 \frac{d\theta}{|\rho_\theta(x - y)|^t} = \infty \quad (3.10)$$

for any $t > 1/2$. Now, if the whole set $B \subset \mathbb{R}^3$ is contained on the line perpendicular to $\gamma(\theta_0)$ and $\dot{\gamma}(\theta_0)$, then all the differences $x - y$, $x, y \in B$ enjoy the same property. Thus,

$$\int_0^1 I_t(\rho_{\theta\sharp}\mu) d\theta = \infty$$

for any $t > 1/2$ and for any Borel measure μ supported on B .

For the projections π_θ , the situation is not so clear-cut. Again, the direct potential theoretic approach fails, because if $x - y \in \ell_{\theta_0} = V_{\theta_0}^\perp$ for some $\theta_0 \in (0, 1)$, then (3.10) holds for any $t > 1$, with ρ_θ replaced by π_θ . But, this time, we do not know if one can construct a set $B \subset \mathbb{R}^3$ with $\dim_{\mathbb{H}} B > 1$ such that most of the differences $x - y$, $x, y \in B$, lie on the lines ℓ_θ , $\theta \in [0, 1]$. Thus, for all we know, it is still possible that estimates of the form

$$\int_0^1 I_s(\pi_{\theta\sharp}\mu) d\theta \lesssim I_t(\mu) < \infty \quad (3.11)$$

hold for $1 < s < t$ and for suitable chosen measures μ supported on B .

3.2.1. Proof of Theorem 1.6(b): a sketch. Being unable to verify an estimate of the form (3.11) – and knowing its impossibility for projections onto lines – our proof takes a different road. We will now give a heuristic outline of the argument, before working out the technical details. We with the counter assumption that the dimension of the projections $\pi_\theta(B)$ drops very close to one in positively many directions $\theta \in [0, 1]$. Using this and the non-degeneracy condition, we find two short, disjoint, compact subintervals $I, J \subset [0, 1]$ with the following properties:

- (i) The dimension of the projections $\pi_\theta(B)$ is very close to one for ‘almost all’ parameters $\theta \in I \cup J$.
- (ii) The surface

$$C_I := \bigcup_{\theta \in I} \ell_\theta$$

is ‘directionally separated’ from the lines ℓ_θ , $\theta \in J$, in the sense that if $x, y \in C_I$, then $x - y$ forms a large angle with any such line ℓ_θ .

The next step is to project the set B onto the planes V_θ , $\theta \in I$. Because of (i), we know that the projections π_θ are, on average, far from bi-Lipschitz. This implies the existence of many differences near the lines $V_\theta^\perp = \ell_\theta$, $\theta \in I$. Building on

this information, we find a large subset $\tilde{B} \subset B$ lying entirely in a small neighbourhood of C_I . The closer the dimension of the projections $\pi_\theta(B)$ drops to one for $\theta \in I$, the larger we can choose \tilde{B} . Then, we recall (ii) and observe that the differences $x - y$ with $x, y \in \tilde{B}$ are directionally far from the lines ℓ_θ , $\theta \in J$ (at least if $|x - y|$ is large enough). This means, essentially, that the restrictions $\pi_\theta|_{\tilde{B}}: \tilde{B} \rightarrow \mathbb{R}^2$, $\theta \in J$, are bi-Lipschitz and shows that the dimension of $\pi_\theta(B)$ exceeds the dimension of \tilde{B} for $\theta \in J$. If the dimension of \tilde{B} was taken close enough to the dimension of B , we end up contradicting (i).

3.3. Proof of Theorem 1.6: the details. We will not discuss the proof of Theorem 1.6(a) informally, since the general outline resembles so closely the one in the proof of Theorem 1.6(b). Our first aim is to reduce the proof of Theorem 1.6 to verifying a discrete statement, Theorem 3.18, which concerns sets and projections at a single scale $\delta > 0$. To this end, we need some definitions.

Definition 3.12 ((δ, s) -sets). Let $\delta, s > 0$, and let $P \subset \mathbb{R}^3$ be a finite δ -separated set. We say that P is a (δ, s) -set, if it satisfies the estimate

$$|P \cap B(x, r)| \lesssim \left(\frac{r}{\delta}\right)^s, \quad x \in \mathbb{R}^3, r \geq \delta.$$

Here $|\cdot|$ refers to cardinality, but, in the sequel, it will also be used to denote length in \mathbb{R} and area in \mathbb{R}^2 . This should cause no confusion, since for any set A only one of the meanings of $|A|$ makes sense. The δ -neighbourhood of P will often be denoted by $B := P(\delta)$.

In a way to be quantified in the next lemma, (δ, s) -sets are well-separated δ -nets inside sets with positive s -dimensional Hausdorff content (denoted by \mathcal{H}_∞^s). This principle – a discrete Frostman’s lemma – is most likely folklore, but we could not locate a reference for exactly the formulation we need. So, we choose to include a proof in Appendix A.

Lemma 3.13 (Frostman). Let $\delta, s > 0$, and let $B \subset \mathbb{R}^3$ be any set with $\mathcal{H}_\infty^s(B) =: \kappa > 0$. Then there exists a (δ, s) -set $P \subset B$ with cardinality $|P| \gtrsim \kappa \cdot \delta^{-s}$.

As we have seen, the potential theoretic method cannot be used to improve Proposition 1.4, because the projections onto planes (resp. lines) may have first (resp. second) order zeros. Such zeros lie on certain ‘bad lines’, the unions of which form ‘bad cones’ in \mathbb{R}^3 . Let us establish notation for these objects.

Definition 3.14 (Cones spanned by curves on S^2). Let $\gamma: [0, 1] \rightarrow S^2$ be a curve. If $I \subset [0, 1]$ is compact subinterval, we write

$$C_I(\gamma) := \bigcup_{\theta \in I} \text{span}(\gamma(\theta)) \subset \mathbb{R}^3.$$

Two special cases of this definition are of particular interest:

Definition 3.15 (Bad lines and bad cones for projection families). Let $\gamma: U \rightarrow S^2$ be a non-degenerate curve as in Definition 1.1, and let $\eta: U \rightarrow S^2$ be the curve

$$\eta(\theta) := \frac{\gamma(\theta) \times \dot{\gamma}(\theta)}{|\gamma(\theta) \times \dot{\gamma}(\theta)|}.$$

(a) A *bad line* for the projection family $(\rho_\theta)_{\theta \in U}$ is any line of the form

$$b_\theta := \text{span}(\eta(\theta)) \subset \mathbb{R}^3, \quad \theta \in U.$$

Unions of bad lines form *bad cones*, as in the previous definition: if $I \subset [0, 1]$ is a compact subinterval, we write

$$C_I^\rho := C_I(\eta).$$

(b) For the projection family $(\pi_\theta)_{\theta \in U}$, the *bad lines* have the form

$$\ell_\theta = \text{span}(\gamma(\theta)).$$

We also define the *bad cones*

$$C_I^\pi := C_I(\gamma), \quad I \subset [0, 1].$$

The definitions of bad lines and cones for $(\rho_\theta)_{\theta \in U}$ and $(\pi_\theta)_{\theta \in U}$ are closely related with the zeroes of the projections. For instance,

$$x \in b_{\theta_0} \iff x \perp \gamma(\theta_0) \text{ and } x \perp \dot{\gamma}(\theta_0),$$

where the right hand side is just another way of saying that

$$\theta \mapsto \rho_\theta(x) = \gamma(\theta) \cdot x \quad \text{and} \quad \theta \mapsto \partial_\theta \rho_\theta(x) = \dot{\gamma}(\theta) \cdot x$$

vanish simultaneously at $\theta = \theta_0$. In particular, if $x \in b_{\theta_0}$, then

$$\int_{\theta_0 - \varepsilon}^{\theta_0 + \varepsilon} \frac{d\theta}{|\rho_\theta(x)|^t} = \infty$$

for any $\varepsilon > 0$ and $t > 1/2$. For the projections π_θ , the situation is even simpler: the mapping $\theta \mapsto |\pi_\theta(x)|$ has a (first order) zero at $\theta = \theta_0$, if and only if $x \in \ell_{\theta_0}$.

Now we can explain how the non-degeneracy hypothesis (1.2) is used in the proof of Theorem 1.6. It will ensure that if $I, J \subset [0, 1]$ are appropriately chosen short intervals, then the bad cones C_I^ρ, C_J^ρ (in part (a)) or C_I^π, C_J^π (in part (b)) ‘point in essentially different directions’. This concept is captured by the next definition:

Definition 3.16. Let $\gamma: [0, 1] \rightarrow S^2$ be a any curve, and let $I, J \subset [0, 1]$ be disjoint compact subintervals. We write

$$C_I(\gamma) \nparallel C_J(\gamma),$$

if there is a constant $c = c(\gamma, I, J) > 0$ with the following property. If $x, y \in C_I(\gamma)$ and $\xi \in C_J(\gamma) \cap S^2 = \{\gamma(\theta) : \theta \in J\}$, then

$$\left| \frac{x - y}{|x - y|} - \xi \right| \geq c.$$

An equivalent way to state the condition is to say that there is a constant $L = L(\gamma, I, J) < 1$ such that every orthogonal projection from $C_I(\gamma)$ to a line on $C_J(\gamma)$ is L -Lipschitz. The next lemma shows how to find intervals $I, J \subset [0, 1]$ such that $C_I(\gamma) \nparallel C_J(\gamma)$.

Lemma 3.17. *Given a C^2 curve $\gamma : [0, 1] \rightarrow S^2$ with nowhere vanishing tangent, suppose that $\theta_1, \theta_2 \in (0, 1)$ are such that $\gamma(\theta_2) \notin \text{span}(\{\gamma(\theta_1), \dot{\gamma}(\theta_1)\})$. Then there exist $\varepsilon_1, \varepsilon_2 > 0$ such that*

$$C_I \nparallel C_J$$

with $I = [\theta_1 - \varepsilon_1, \theta_1 + \varepsilon_1]$, $J = [\theta_2 - \varepsilon_2, \theta_2 + \varepsilon_2]$, $C_I = C_I(\gamma)$ and $C_J = C_J(\gamma)$.

This result is rather intuitive; a rigorous proof is given in Appendix B. Now we are prepared to formulate a δ -discretised version of Theorem 1.6.

Theorem 3.18. *Let $s > 0$, and let $P \subset B(0, 1)$ be a (δ, s) -set with cardinality $|P| \sim \delta^{-s}$. Write $B := P(\delta)$. The following statements hold for $\delta > 0$ small enough.*

(a) *Suppose that $I, J \subset [0, 1]$ are intervals such that*

$$C_I^\rho \nparallel C_J^\rho.$$

If $s > 1/2$, there exists $\varepsilon_1 = \varepsilon_1(s) > 0$ and $\sigma_1 = \sigma_1(s) > 1/2$ with the following property. Suppose that $E_I \subset I$ and $E_J \subset J$ have lengths $|E_I| \geq \delta^{\varepsilon_1}$ and $|E_J| \geq \delta^{\varepsilon_1}$. Then there exists a direction $\theta \in E_I \cup E_J$ such that

$$|\rho_\theta(B)| \geq \delta^{1-\sigma_1}.$$

(b) *Suppose that $I, J \subset [0, 1]$ are intervals such that*

$$C_I^\pi \nparallel C_J^\pi.$$

If $s > 1$, there exists $\varepsilon_2 = \varepsilon_2(s) > 0$ and $\sigma_2 = \sigma_2(s) > 1$ with the following property. Suppose that $E_I \subset I$ and $E_J \subset J$ have lengths $|E_I| \geq \delta^{\varepsilon_2}$ and $|E_J| \geq \delta^{\varepsilon_2}$. Then there exists a direction $\theta \in E_I \cup E_J$ such that

$$|\pi_\theta(B)| \geq \delta^{2-\sigma_2}.$$

Let us briefly explain how Theorem 1.6 follows from its δ -discretised variant. First, we note that in order to derive statements like Theorem 1.6 for the packing dimension of projections, it suffices to prove their analogues for the *upper box dimension* $\overline{\dim}_B$, defined by

$$\overline{\dim}_B R = \limsup_{\delta \rightarrow 0} \frac{\log N(R, \delta)}{-\log \delta}$$

for bounded sets $R \subset \mathbb{R}^d$, where $N(R, \delta)$ is the least number of balls of radius δ required to cover R . This reduction is possible thanks to the following lemma.

Lemma 3.19. *Let $\sigma > 0$, let μ be a Borel regular measure, and let $B \subset \mathbb{R}^3$ be a μ -measurable set such that $\mu(B) > 0$, and*

$$|\{\theta \in [0, 1] : \dim_p \rho_\theta(B) < \sigma\}| > 0.$$

Then there exists a compact set $K \subset B$ with $\mu(K) > 0$ such that

$$|\{\theta \in [0, 1] : \overline{\dim}_B \rho_\theta(K) < \sigma\}| > 0.$$

Proof. The proof is the same as that of [13, Lemma 4.5], except for some obvious changes in notation. \square

The statement also holds with the projections ρ_θ replaced by π_θ . Now, if Theorem 1.6 failed for \dim_p , there would exist an analytic set $B \subset \mathbb{R}^3$ with $\dim_H B > s$ such that the projections of B have packing dimension less than $\sigma \in \{\sigma_1, \sigma_2\}$ in positively many directions. Then, we could find a Frostman measure μ inside B and apply Lemma 3.19 to B , μ and σ . The conclusion would be that also the $\overline{\dim}_B$ -variant of Theorem 1.6 has to fail in positively many directions.

Proof of Theorem 1.6. We will now describe how to use Theorem 3.18(a) to prove the $\overline{\dim}_B$ -variant of Theorem 1.6(a). The deduction of Theorem 1.6(b) from Theorem 3.18(b) is analogous. To reach a contradiction, suppose that $\dim_H B = s > 1/2$, but there is a positive length subset $E \subset [0, 1]$ such that

$$\overline{\dim}_B \rho_\theta(B) < \sigma_1 - c \quad (3.20)$$

for all $\theta \in E$ and some small constant $c > 0$. Here $\sigma_1 = \sigma_1(s) > 1/2$ is the constant from Theorem 3.18. Fix a Lebesgue point $\theta_1 \in E$, and then choose another Lebesgue point $\theta_2 \in E$ such that

$$\eta(\theta_2) \notin \text{span}(\{\eta(\theta_1), \dot{\eta}(\theta_1)\}), \quad (3.21)$$

where $\eta = \gamma \times \dot{\gamma}/|\gamma \times \dot{\gamma}|$. This is precisely where we need the non-degeneracy hypothesis.

Lemma 3.22. *Let $\gamma : U \rightarrow S^2$ be a C^3 curve satisfying the non-degeneracy condition (1.2). Then the curve $\eta : U \rightarrow S^2$, given by $\eta := \frac{\gamma \times \dot{\gamma}}{|\gamma \times \dot{\gamma}|}$, fulfills the same condition, that is,*

$$\text{span}\{\eta(\theta), \dot{\eta}(\theta), \ddot{\eta}(\theta)\} = \mathbb{R}^3, \quad (3.23)$$

for every $\theta \in U$.

It follows from this lemma, which is proved in Appendix B, and from Lemma 3.4 that for any given 2-plane $W \subset \mathbb{R}^2$ there are only finitely many choices of $\theta_2 \in [0, 1]$ such that $\eta(\theta_2) \in W$: indeed, if \bar{n} is the normal vector of the plane W , Lemma 3.4 implies that the mapping $\theta \mapsto \eta(\theta) \cdot \bar{n}$ can only have a bounded number of zeros $\theta \in [0, 1]$. Now we may apply Lemma 3.17 to the path η : thus, we find disjoint compact intervals $I \ni \theta_1$ and $J \ni \theta_2$ with the property that

$$C_I^\rho \not\parallel C_J^\rho.$$

This places us in a situation, where we can apply Theorem 3.18(a). Let $\varepsilon_1 > 0$ be the number defined there, and let $\delta > 0$ be so small the lengths of $E_I := E \cap I$ and $E_J := E \cap J$ exceed δ^{ε_1} . Then use Lemma 3.13 to find a (δ, s) -set $P \subset B$ with cardinality $|P| \sim \delta^{-s}$. From (3.20), we see that

$$|\rho_\theta(P(\delta))| \leq |\rho_\theta(B(\delta))| \lesssim \delta^{-\sigma_1+c}$$

for $\delta > 0$ and $\theta \in E_I \cup E_J$. For $\delta > 0$ small enough, this is incompatible with the conclusion of Theorem 3.18(a). \square

It remains to prove Theorem 3.18. The basic approach for both (a) and (b) is the same, but (b) is slightly simpler from a technical point of view. This is why we choose to give the proof of (b) first.

Proof of Theorem 3.18(b). Recall that $P \subset B(0, 1)$ is a (δ, s) -set of cardinality $P \sim \delta^{-s}$, and $B = P(\delta)$. We make the counter assumption that

$$|\pi_\theta(B)| < \delta^{2-\sigma_2} \quad (3.24)$$

for all $\theta \in E_I \cup E_J$. The constant $\sigma_2 \in (1, s)$ will be fixed as the proof progresses. In particular, (3.24) means that for $\theta \in E_J$, the projection $\pi_\theta(P)$ can be covered by $\lesssim \delta^{-\sigma_2}$ discs of radius $\delta > 0$.

For $x, y \in \mathbb{R}^3$ and $\theta \in E_I \cup E_J$, we define the relation $x \sim_\theta y$ as follows:

$$x \sim_\theta y \iff x \neq y \text{ and } |\pi_\theta(x) - \pi_\theta(y)| \leq \delta,$$

We also write

$$T_I(x, y) := |\{\theta \in E_I : x \sim_\theta y\}|.$$

Our first aim is to use (3.24) to find a lower bound for the quantity

$$\mathcal{E} := \sum_{x, y \in P} T_I(x, y) = \int_{E_I} |\{(x, y) \in P^2 : x \sim_\theta y\}| d\theta$$

Fix $\theta \in E_I$ and choose a minimal (in terms of cardinality) collection of disjoint discs $D_1, \dots, D_{N(\theta)} \subset \mathbb{R}^2$ such that $\text{diam}(D_j) = \delta$ and

$$\left| P \cap \bigcup_{j=1}^{N(\theta)} \pi_\theta^{-1}(D_j) \right| \gtrsim |P| \sim \delta^{-s}. \quad (3.25)$$

Then (3.24) implies that $N(\theta) \lesssim \delta^{-\sigma_2}$. Next, we discard all the discs D_j such that $|P \cap \pi_\theta^{-1}(D_j)| \leq 1$. This way only $\lesssim \delta^{-\sigma_2}$ points are deleted from the left hand side of (3.25), so the inequality remains valid for the remaining collection of discs, and for small $\delta > 0$. The point of the discarding process is simply to ensure that

$$|\{(x, y) \in [P \cap \pi_\theta^{-1}(D_j)]^2 : x \sim_\theta y\}| = |P \cap \pi_\theta^{-1}(D_j)|^2 - |P \cap \pi_\theta^{-1}(D_j)| \gtrsim |P \cap \pi_\theta^{-1}(D_j)|^2$$

for the remaining discs D_j . This in mind, we estimate \mathcal{E} from below:

$$\begin{aligned} \mathcal{E} &\gtrsim \int_{E_I} \sum_{\text{remaining } D_j} |P \cap \pi_\theta^{-1}(D_j)|^2 d\theta \\ &\geq \int_{E_I} \frac{1}{N(\theta)} \left(\sum_{\text{remaining } D_j} |P \cap \pi_\theta^{-1}(D_j)| \right)^2 d\theta \\ &\gtrsim |E_I| \cdot \delta^{\sigma_2} \cdot |P|^2 \gtrsim \delta^{\sigma_2 + \varepsilon_2 - 2s}. \end{aligned} \quad (3.26)$$

Our second aim is to show that (3.26) gives some structural information about P , if ε_2 and σ_2 are small. For $x \in P$ we define a ‘neighbourhood’ $N(x)$ of x by

$$N(x) := P \cap (x + C_I^\pi(2\delta)).$$

Recall that C_I^π was defined as the union of the lines ℓ_θ , $\theta \in I$, perpendicular to the planes V_θ . The reason for defining $N(x)$ as we do is the following: if $y \in P \setminus N(x)$, then $y - x \notin C_I^\pi(2\delta)$, so that the difference $y - x$ stays at distance $> \delta$ from any of the orthogonal complements of the planes V_θ . In particular, it is not possible that $x \sim_\theta y$ for any parameter $\theta \in I$, which implies that

$$\mathcal{E} = \sum_{x \in P} \sum_{y \in N(x)} T_I(x, y). \quad (3.27)$$

To connect the sizes of the neighbourhoods $N(x)$ with (3.26), we need a universal estimate for $T_I(x, y)$:

Lemma 3.28. *Let $x, y \in \mathbb{R}^3$ be δ -separated points. Then*

$$T_I(x, y) \lesssim \frac{\delta}{|x - y|}.$$

Proof. Apply the sub-level estimate (3.9) with $\lambda = \delta$. □

Now we are equipped to search for a large set $N(x) \subset P$. Suppose that $|N(x)| \leq \delta^{-s+\varepsilon}$ for every $x \in P$, where

$$\varepsilon = \frac{s(s-1)}{2s-1}.$$

Write $A_j(x) := \{y \in \mathbb{R}^3 : 2^j \leq |y - x| \leq 2^{j+1}\}$. Recalling (3.27) and using the inequality

$$\min\{a, b\} \leq a^{1-\frac{1}{s}} b^{\frac{1}{s}}, \quad a, b \geq 0,$$

we estimate as follows:

$$\begin{aligned} \mathcal{E} &= \sum_{x \in P} \sum_{\delta \leq 2^j \leq 1} \sum_{y \in A_j(x) \cap N(x)} T_I(x, y) \\ &\lesssim \delta \sum_{x \in P} \sum_{\delta \leq 2^j \leq 1} 2^{-j} \min\{|N(x)|, |P \cap B(x, 2^{j+1})|\} \\ &\lesssim \delta \sum_{x \in P} \sum_{\delta \leq 2^j \leq 1} 2^{-j} \min\left\{\delta^{-s+\varepsilon}, \left(\frac{2^j}{\delta}\right)^s\right\} \\ &\leq \sum_{x \in P} \sum_{\delta \leq 2^j \leq 1} \delta^{(-s+\varepsilon)(1-\frac{1}{s})} \sim \delta^{-2s+\varepsilon(1-\frac{1}{s})+1} \cdot \log\left(\frac{1}{\delta}\right). \end{aligned}$$

So, assuming that $|N(x)| \leq \delta^{-s+\varepsilon}$, we can combine the bound above with (3.26) to conclude that

$$\delta^{\sigma_2+\varepsilon_2-2s} \lesssim \delta^{-2s+\varepsilon(1-\frac{1}{s})+1} \cdot \log\left(\frac{1}{\delta}\right).$$

Since the implicit constants are independent of $\delta > 0$, this shows that either

- (i) There exists a point $x \in P$ with $|N(x)| \geq \delta^{-s+\varepsilon}$, or
- (ii) $\sigma_2 + \varepsilon_2 \geq 1 + \varepsilon(1 - \frac{1}{s})$.

The proof of Theorem 3.18(b) nears its end. Our next lemma will show that the projections $\pi_\theta|_{N(x)}$, $\theta \in J$, are essentially bi-Lipschitz, so the counter assumption $|\pi_\theta(B)| < \delta^{2-\sigma_2}$ for $\theta \in J$ will force the inequality $|N(x)| \lesssim \delta^{-\sigma_2}$. In case (i) holds, this shows that

$$\sigma_2 \geq s - \varepsilon = 1 + \frac{(s-1)^2}{2s-1}.$$

If (i) fails, we conclude from (ii) that

$$\sigma_2 + \varepsilon_2 \geq 1 + \varepsilon(1 - \frac{1}{s}) = 1 + \frac{(s-1)^2}{2s-1}. \quad (3.29)$$

Either way, Theorem 3.18(b) is true for any pair $(\sigma_2, \varepsilon_2)$ satisfying (3.29).

It remains to state and prove the bi-Lipschitz lemma. In order to make the same lemma useful in the proof of Theorem 3.18(a), we state a slightly more general version than we would need here.

Lemma 3.30. *Assume that $\gamma: [0, 1] \rightarrow S^2$ is a curve, and $C_I \nparallel C_J$ for some intervals $I, J \subset [0, 1]$, where $C_I = C_I(\gamma)$ and $C_J = C_J(\gamma)$. Let $x \in \mathbb{R}^3$, $\tau > 0$. Then, there exists a constant $C \geq 1$, depending only on γ , I and J , such that whenever $y, y' \in B(0, 1)$ satisfy*

$$y, y' \in x + C_I(\delta^\tau) \quad \text{and} \quad |y - y'| \geq C\delta^\tau,$$

then

$$\left| \frac{y - y'}{|y - y'|} - \xi \right| \gtrsim_{\gamma, I, J} 1, \quad \xi \in C_J \cap S^2. \quad (3.31)$$

Proof. Let $c > 0$ be the constant from the definition of $C_I(\gamma) \nparallel C_J(\gamma)$: thus, if $u, v \in x + C_I$, then

$$\left| \frac{u - v}{|u - v|} - \xi \right| \geq c$$

for any vector $\xi \in C_J \cap S^2$. Suppose that $y, y' \in B(0, 1)$ satisfy the hypotheses of the lemma, and find $y_0, y'_0 \in x + C_I$ such that $|y - y_0| \leq \delta^\tau$ and $|y' - y'_0| \leq \delta^\tau$. Note that the points y_0 and y'_0 are at least $C\delta^\tau/2$ apart for $C \geq 10$, and

$$\left| \frac{y - y'}{|y - y'|} - \xi \right| \geq c - \left| \frac{y_0 - y'_0}{|y_0 - y'_0|} - \frac{y - y'}{|y - y'|} \right|, \quad \xi \in C_J \cap S^2. \quad (3.32)$$

To estimate the negative term, consider the mapping $b: \mathbb{R}^3 \setminus B(0, C\delta^\tau/2) \rightarrow S^2$, defined by $b(x) = x/|x|$. Choosing C large enough, the mapping b can be made L -Lipschitz with $L \leq c\delta^{-\tau}/4$, so

$$\begin{aligned} \left| \frac{y_0 - y'_0}{|y_0 - y'_0|} - \frac{y - y'}{|y - y'|} \right| &= |b(y_0 - y'_0) - b(y - y')| \\ &\leq \frac{c\delta^{-\tau}}{4}(|y - y_0| + |y' - y'_0|) \leq \frac{c}{2}. \end{aligned}$$

This and (3.32) give (3.31). \square

In the proof of Theorem 3.18(b), we apply the lemma with the non-parallel bad cones C_I^π and C_J^π and with $\tau = 1$: let $C \gtrsim_{\gamma, I, J}$ be the constant appearing in the statement of the lemma. If option (i) above is realised, we choose a $C\delta$ -net $\tilde{P} \subset N(x)$. Then $|\tilde{P}| \gtrsim \delta^{-s+\varepsilon}$, and the angle between any difference $y - y'$, $y, y' \in \tilde{P}$, and any line $\ell_\theta = V_\theta^\perp$, $\theta \in J$, is bounded from below by a constant. This means that the restrictions $\pi_\theta|_{\tilde{P}}$ are bi-Lipschitz, so $|\pi_\theta(\tilde{P}(\delta))| \gtrsim \delta^{2-s+\varepsilon}$ for any $\theta \in J$. The proof of Theorem 3.18(b) is now completed in the manner we described above. \square

Next, we turn to the proof of Theorem 3.18(a). The structure will be familiar, but there are some additional steps to take.

Proof of Theorem 3.18(a). All the way down to the lower energy estimate (3.26) the argument follows the proof of Theorem 1.6(b) with the obvious changes

$$\pi_\theta \rightsquigarrow \rho_\theta, \quad \varepsilon_2 \rightsquigarrow \varepsilon_1, \quad \sigma_2 \rightsquigarrow \sigma_1,$$

and choosing the sets $D_1, \dots, D_{N(\theta)}$ as δ -intervals in \mathbb{R} rather than δ -discs in \mathbb{R}^2 . The analogue of (3.27) is

$$\mathcal{E} \gtrsim \delta^{\sigma_1 + \varepsilon_1 - 2s}. \quad (3.33)$$

The first essential difference appears in the definition of the ‘neighbourhoods’ $N(x)$, $x \in P$. This time

$$N(x) := P \cap (x + C_I^\rho(\delta^\tau)),$$

where $\tau \in (0, 1/2)$ is a parameter to be chosen soon. Recall that C_I^ρ is the union of the bad lines b_θ , $\theta \in I$, spanned by the vectors $\gamma(\theta) \times \dot{\gamma}(\theta)$. This time, if a point $y \in P$ stays away from a neighbourhood $N(x)$, we may **not** conclude that $x \not\sim_\theta y$ for all $\theta \in I$. Instead, the event $y \notin N(x)$ signifies that the mapping $\theta \mapsto \rho_\theta(x - y)$ does not have a second order zero on the interval I . Consequently, we have an improved estimate for $T_I(x, y)$. An ‘improved estimate’ means an improvement over the following universal bound, analogous to the one in Lemma 3.28:

Lemma 3.34. *Let $x, y \in \mathbb{R}^3$ be δ -separated points. Then*

$$T_I(x, y) \lesssim \left(\frac{\delta}{|x - y|} \right)^{1/2}.$$

Proof. Apply the sub-level estimate (3.6) with $\lambda = \delta$. \square

Lemma 3.35. *Suppose that $0 \leq \tau < 1$, and $x, y \in \mathbb{R}^3$ satisfy*

$$y - x \notin C_I^\rho(\delta^\tau).$$

Then

$$T_I(x, y) \lesssim \delta^{1-\tau}.$$

Proof. The condition $y - x \notin C_I^\rho(\delta^\tau)$ is another way of saying that $d(y - x, b_\theta) \geq \delta^\tau$ for all bad lines $b_\theta = \text{span}\{\gamma(\theta) \times \dot{\gamma}(\theta)\} \subset C_I^\rho$. Now, note that the distance of a vector z from b_θ equals the length of the projection $\tilde{\pi}_\theta(z)$ of z onto the plane $b_\theta^\perp = \text{span}(\{\gamma(\theta), \dot{\gamma}(\theta)\})$. Hence

$$\left[((x - y) \cdot \gamma(\theta))^2 + \left((x - y) \cdot \frac{\dot{\gamma}(\theta)}{|\dot{\gamma}(\theta)|} \right)^2 \right]^{1/2} = |\tilde{\pi}_\theta(x - y)| \geq \delta^\tau.$$

Since $|\dot{\gamma}(\theta)|$ is bounded from below on I , and $\tau < 1$, we may infer that

$$|\rho_\theta(x - y)| \leq 2\delta \implies |\partial_\theta \rho_\theta(x - y)| \gtrsim \delta^\tau.$$

This implies that the set $\{\theta \in I : |\rho_\theta(x - y)| < 2\delta\}$ consists of intervals I_1, \dots, I_N around the zeros of $\theta \mapsto \rho_\theta(x - y)$ on I , and possibly two intervals having a common endpoint with I . We saw in Lemma 3.4 that the number of zeros of $\theta \mapsto \rho_\theta(x - y)$, $x \neq y$, on any compact subinterval of U is bounded by a constant independent of $x - y$. So, in order to estimate the length of $\{\theta \in I : |\rho_\theta(x - y)| < 2\delta\}$ – and the cardinality of $T_I(x, y)$ – it suffices to bound the lengths of the intervals I_i . But the lower bound of the derivative $\partial_\theta \rho_\theta(x - y)$ readily shows that

$$|I_i| \lesssim \delta^{1-\tau},$$

which completes the proof of the lemma. \square

Next, as in the proof of Theorem 3.18(b), we claim that the lower bound (3.33) forces a dichotomy: either ε_1 and σ_1 are large, or there exists a neighbourhood $N(x)$ with cardinality $|N(x)| \geq \delta^{-s+\varepsilon_I}$. Here

$$\varepsilon_I = \frac{s(2s-1)}{12s^2+4s-1} - \kappa,$$

where $\kappa > 0$ is arbitrary (but so small that $\varepsilon_I > 0$). Let us first estimate \mathcal{E} from above, **assuming** $|N(x)| \leq \delta^{-s+\varepsilon_I}$ for every $x \in P$:

$$\mathcal{E} = \sum_{x \in P} \sum_{y \in N(x)} T_I(x, y) + \sum_{x \in P} \sum_{y \in P \setminus N(x)} T_I(x, y) =: S_1 + S_2.$$

The sum S_1 is bounded using the universal bound in Lemma 3.34, combined with the size estimate for $|N(x)|$:

$$\begin{aligned} S_1 &= \delta^{1/2} \sum_{x \in P} \sum_{\delta \leq 2^j \leq 1} 2^{-j/2} \min\{|N(x)|, |P \cap B(x, 2^{j+1})|\} \\ &\lesssim \delta^{1/2} \sum_{x \in P} \sum_{\delta \leq 2^j \leq 1} 2^{-j/2} \min\left\{\delta^{-s+\varepsilon_I}, \left(\frac{2^j}{\delta}\right)^s\right\} \\ &\leq \sum_{x \in P} \sum_{\delta \leq 2^j \leq 1} \delta^{(-s+\varepsilon_I)(1-\frac{1}{2s})} \sim \delta^{-2s+\varepsilon_I(1-\frac{1}{2s})+1/2} \cdot \log\left(\frac{1}{\delta}\right). \end{aligned}$$

To estimate S_2 , we set

$$\tau = 1/2 - (\varepsilon_I + \varepsilon_J)(1 - \frac{1}{2s}) > 0,$$

where

$$\varepsilon_J := \frac{4s^2}{12s^2 + 4s - 1} > 0,$$

and apply Lemma 3.35 with this particular choice of τ :

$$S_2 \leq \sum_{x \in P} \sum_{y \in P \setminus N(x)} \delta^{1-\tau} \lesssim \delta^{-2s+(\varepsilon_I+\varepsilon_J)(1-\frac{1}{2s})+1/2}.$$

With our choices of parameters, we see that $\mathcal{E} = S_1 + S_2 \lesssim S_1$. Comparing the upper bound for S_1 with (3.33), we conclude that either one of the following options must hold:

- (i) There exists $x \in P$ with $|N(x)| \geq \delta^{-s+\varepsilon_I}$, or
- (ii) $\sigma_1 + \varepsilon_1 \geq 1/2 + \varepsilon_I(1 - \frac{1}{2s})$.

Assuming momentarily that (i) holds, we will now start using the information about the size of the projections $\rho_\theta(B)$, $\theta \in J$. Write $\tilde{P} := N(x)$, where $|N(x)| \geq \delta^{-s+\varepsilon_I}$. Also, write $\tilde{B} := \tilde{P}(\delta)$. Since $\tilde{P} \subset P$, we know that $|\rho_\theta(\tilde{B})| \leq \delta^{1-\sigma_1}$ for $\theta \in E_J$. So, if we set

$$T_J := |\{\theta \in E_J : x \sim_\theta y\}|$$

and define

$$\mathcal{E}_J := \sum_{x, y \in \tilde{P}} T_J(x, y),$$

the same argument that gave (3.33) yields the lower bound

$$\mathcal{E}_J \gtrsim \delta^{\sigma_2+\varepsilon_2+2\varepsilon_I-2s}. \quad (3.36)$$

With this in mind, we set hunting for a large neighbourhood

$$N_J(y) := \tilde{P} \cap (y + C_J^\rho(\delta^\tau)), \quad y \in \tilde{P}.$$

The parameter $\tau > 0$ is the same as before. If all such neighbourhoods have size $|N_J(y)| \leq \delta^{-s+\varepsilon_I+\varepsilon_J}$, precisely the same argument as above shows that

$$\mathcal{E}_J \lesssim \delta^{-2s+(\varepsilon_I+\varepsilon_J)(1-\frac{1}{2s})+1/2} \cdot \log\left(\frac{1}{\delta}\right). \quad (3.37)$$

Indeed, one only needs to observe that the bounds for $T_I(x, y)$ in Lemmas 3.34 and 3.35 transfer without change to bounds for $T_J(x, y)$. Comparing (3.36) and (3.37), we arrive at a familiar alternative:

- (i') There exists $y \in \tilde{P}$ with $|N_J(y)| \geq \delta^{-s+\varepsilon_I+\varepsilon_J}$, or
- (ii') $\sigma_1 + \varepsilon_1 \geq 1/2 + (\varepsilon_I + \varepsilon_J)(1 - \frac{1}{2s}) - 2\varepsilon_I$.

Now the proof is nearly complete. The next step is to show that (i) and (i') are mutually incompatible with our choices of ε_I and ε_J . Consequently, from our alternatives, we will see that either (ii) holds, or (i) **and** (ii') hold. Both options will lead to a lower bound for σ_1 .

To establish the incompatibility of (i) and (i'), we apply Lemma 3.30 with τ and the non-parallel cones C_I^ρ, C_J^ρ . Let $C \gtrsim_{\gamma, I, J} 1$ be a constant, which appears in the lemma with these parameters. Assuming (i) and (i'), recalling the formulae for ε_I and ε_J , and using the fact that $N_J(y)$ is a (δ, s) -set, we have

$$\begin{aligned} |N_J(y) \cap B(y, C\delta^\tau)| &\lesssim_C \left(\frac{\delta^\tau}{\delta}\right)^s = \delta^{(\tau-1)s} \\ &= \delta^{-s/2 - (\varepsilon_I + \varepsilon_J)(s-1/2)} \\ &= \delta^{\kappa(s+1/2)} \cdot \delta^{-s + \varepsilon_I + \varepsilon_J} \\ &\leq \delta^{\kappa(s+1/2)} \cdot |N_J(y)|. \end{aligned}$$

This shows that no matter how large C is, for small enough $\delta > 0$ the set $N_J(y)$ cannot be contained in the ball $B(y, C\delta^\tau)$. So, if (i) and (i') hold, and $\delta > 0$ is small enough, we can find a point

$$y' \in N_J(y) \subset \tilde{P} \subset x + C_I^\rho(\delta^\tau)$$

with

$$|y - y'| > C\delta^\tau. \quad (3.38)$$

We infer from Lemma 3.30 that

$$\left| \frac{y - y'}{|y - y'|} - \xi \right| \gtrsim_{\gamma, I, J} 1, \quad \xi \in C_J^\rho \cap S^2. \quad (3.39)$$

On the other hand, we know that $y' \in N_J(y) \subset y + C_J^\rho(\delta^\tau)$, so there is a line $b_\theta \subset C_J^\rho$, $\theta \in J$, such that $d(y - y', b_\theta) \leq \delta^\tau$. If b_θ is spanned by the unit vector $\xi \in C_J^\rho \cap S^2$, one can combine (3.38) with elementary geometry to show that

$$\left| \frac{y - y'}{|y - y'|} - \xi \right| \lesssim \frac{1}{C},$$

as long as $C \leq \delta^{-\tau}$. This is incompatible with (3.39), if C is large enough (still depending only on γ, I and J). We have established that (i) and (i') cannot hold simultaneously. Thus, if (i) holds, we may infer that also (ii') holds, so $\sigma_1 + \varepsilon_1$ must satisfy the lower bound

$$\sigma_1 + \varepsilon_1 \geq \frac{1}{2} + (\varepsilon_I + \varepsilon_J) \left(1 - \frac{1}{2s}\right) - 2\varepsilon_I = \frac{1}{2} + \frac{1}{2} \cdot \frac{(2s-1)^2}{12s^2 + 4s - 1} + \kappa \left(1 + \frac{1}{2s}\right)$$

But if (i) fails, we know that (ii) holds, and then

$$\sigma_1 + \varepsilon_1 \geq \frac{1}{2} + \varepsilon_I \left(1 - \frac{1}{2s}\right) = \frac{1}{2} + \frac{1}{2} \cdot \frac{(2s-1)^2}{12s^2 + 4s - 1} - \kappa \left(1 - \frac{1}{2s}\right).$$

Either way, making $\kappa > 0$ small, we may choose $\varepsilon_1(s) > 0$ and $\sigma_1(s) > 1/2$ as in Theorem 3.18(a). This completes the proof. \square

4. SETS WITH ADDITIONAL STRUCTURE

This section has two parts. In the first one, we prove Theorem 1.8 for a special class of sets we call *BLP sets*. In the second part, we demonstrate that self-similar sets without rotations are BLP sets, thus concluding the proof of Theorem 1.8.

4.1. BLP sets. We set off with two definitions.

Definition 4.1 (BLP sets). A set $B \subset \mathbb{R}^3$ has the *bi-Lipschitz property*, BLP in short, if for any plane $V \in G(3, 2)$ and $\varepsilon > 0$ there exists a subset $B_{V,\varepsilon} \subset V$ such that

- $\dim_{\text{H}} B_{V,\varepsilon} \geq \dim_{\text{H}} \pi_V(B) - \varepsilon$, and
- the restriction $\pi_V|_{B_{V,\varepsilon}} : B_{V,\varepsilon} \rightarrow V$ is bi-Lipschitz.

Definition 4.2. Let $\ell \in G(3, 1)$. A set $B \subset \mathbb{R}^3$ *stays non-tangentially off the line* ℓ ($B \angle \ell$ for short) if there exists $0 < \alpha < 1$ such that

$$X(0, \ell, \alpha) \cap (B - B) = \emptyset,$$

where

$$X(y, \ell, \alpha) := \{x \in \mathbb{R}^3 : d(x - y, \ell) < \alpha|x - y|\}$$

is a cone with opening angle α around ℓ centered at $y \in \mathbb{R}^3$.

It will be useful to have various reformulations of this property at our disposal.

Lemma 4.3. Let $B \subset \mathbb{R}^3$ and $\ell \in G(3, 1)$. The following properties are equivalent:

- (1) $B \angle \ell$.
- (2) There exists $0 < \alpha < 1$ such that for all $y \in B$ we have $X(y, \ell, \alpha) \cap B = \emptyset$.
- (3) The projection $\pi_V|_B$ onto the plane $V = \ell^\perp$ is bi-Lipschitz with the constant α from the definition of $B \angle \ell$.

Let us now return to the projection family $(\rho_\theta)_{\theta \in U}$. The point of the definitions above is here: if $B \subset \mathbb{R}^3$ is a BLP set, $\theta_0 \in U$ and $\varepsilon > 0$, one may find a subset $B_{\theta_0,\varepsilon} \subset B$ such that $\dim_{\text{H}} B_{\theta_0,\varepsilon} \geq \dim_{\text{H}} \pi_{\theta_0}(B) - \varepsilon$ and $B_{\theta_0,\varepsilon} \angle b_{\theta_0}$, where $b_{\theta_0} \in G(3, 1)$ is the ‘bad line’ spanned by the vector $\gamma(\theta_0) \times \dot{\gamma}(\theta_0)$ and π_{θ_0} is the projection onto $b_{\theta_0}^\perp$. The next proposition explains why this is useful:

Proposition 4.4. Let $B \subset \mathbb{R}^3$ be a set such that $B \angle b_{\theta_0}$ for some $\theta_0 \in U$. Then there exists an open interval $J \ni \theta_0$ such that the family $(\rho_\theta|_B)_{\theta \in J}$ is transversal in the sense of Peres and Schlag, see [14, Definition 2.7].

Proof. Staying ‘non-tangentially off a line’ is an open property in the following sense: if there exists $0 < \alpha < 1$ such that $(B - B) \cap X(0, b_{\theta_0}, \alpha) = \emptyset$, as we assume, then $(B - B) \cap X(0, b_\theta, \alpha/2) = \emptyset$ for θ in a small neighbourhood $J \subset U$ of θ_0 . Now, let $\theta \in J$, and consider the projection π_θ onto the plane $V_\theta = b_\theta^\perp$. According

to Lemma 4.3, the restriction $\pi_\theta|_B$ is bi-Lipschitz with constant $\alpha/2$, which means that

$$\left[((x-y) \cdot \gamma(\theta))^2 + \left((x-y) \cdot \frac{\dot{\gamma}(\theta)}{|\dot{\gamma}(\theta)|} \right)^2 \right]^{1/2} = |\pi_\theta(x-y)| \geq \frac{\alpha}{2}|x-y|$$

for all $x, y \in B$. Taking J short enough, the quantity $|\dot{\gamma}(\theta)|$ is bounded from below by a constant $c > 0$ for $\theta \in J$. Thus, either

$$\left| \rho_\theta \left(\frac{x-y}{|x-y|} \right) \right| \geq \frac{\alpha}{5} \quad \text{or} \quad \left| \partial_\theta \rho_\theta \left(\frac{x-y}{|x-y|} \right) \right| \geq \frac{c\alpha}{5}.$$

for all $x, y \in B, x \neq y$. This means that J is an interval of transversality of order $\beta = 0$ for the projection family $(\rho_\theta|_B)_{\theta \in J}$, in the sense [14, Definition 2.7]. \square

Now we are prepared to prove the analogue of Theorem 1.8 for BLP sets.

Theorem 4.5. *Let $B \subset \mathbb{R}^3$ be a BLP set, and let $(\rho_\theta)_{\theta \in U}$ be a non-degenerate family of projections in the sense of Definition 1.3.*

- (a) *If $0 \leq \dim_{\text{H}} B \leq 1$, then $\dim_{\text{H}} \rho_\theta(B) = \dim_{\text{H}} B$ almost surely.*
- (b) *If $\dim_{\text{H}} B > 1$, and additionally*

$$\dim_{\text{p}} \pi_V(B) = \dim_{\text{H}} \pi_V(B) \tag{4.6}$$

for every plane $V \in G(3, 2)$, then $\rho_\theta(B)$ has positive length almost surely.

Proof. We start with (a). According to Lemma 3.22, the family of lines $(b_\theta)_{\theta \in U}$ is a non-degenerate one. Using Proposition 1.4(b), we see that

$$\dim_{\text{H}} \pi_\theta(B) = \dim_{\text{H}} B \tag{4.7}$$

for almost every $\theta \in U$, where π_θ refers to the projection onto the plane $V_\theta = b_\theta^\perp$. Let $\theta_0 \in U$ be one of the parameters for which (4.7) holds, and fix $\varepsilon > 0$. Since B is a BLP set, we may choose a subset $B_{\theta_0, \varepsilon} \subset B$ such that $\dim_{\text{H}} B_{\theta_0, \varepsilon} \geq \dim_{\text{H}} B - \varepsilon$ and $B_{\theta_0, \varepsilon} \angle b_{\theta_0}$. Then, we infer from Proposition 4.4 that there exists a small interval $J \subset U$ containing θ_0 such that the projections $(\rho_\theta)_{\theta \in J}$ restricted to $B_{\theta_0, \varepsilon}$ are transversal. It follows from [14, Theorem 2.8] that

$$\dim_{\text{H}} \rho_\theta(B) \geq \dim_{\text{H}} \rho_\theta(B_{\theta_0, \varepsilon}) = \dim_{\text{H}} B_{\theta_0, \varepsilon} \geq \dim_{\text{H}} B - \varepsilon$$

for almost every $\theta \in J$. Since (4.7) holds for almost surely, we can run the same argument for almost every $\theta_0 \in U$, proving that $\dim_{\text{H}} \rho_\theta(B) \geq \dim_{\text{H}} B - \varepsilon$ for almost every $\theta \in U$. Letting $\varepsilon \rightarrow 0$ concludes the proof of part (a).

The proof of part (b) is similar, except that this time we resort to Theorem 1.6(b) instead of Proposition 1.4(b). Namely, if $\dim_{\text{H}} B > 1$, we infer from Theorem 1.6(b) and the additional assumption (4.6) that

$$\dim_{\text{H}} \pi_\theta(B) = \dim_{\text{p}} \pi_\theta(B) > 1$$

for almost every $\theta \in U$. Then, fixing almost any $\theta_0 \in U$ and using the BLP property, we find a subset $B_{\theta_0} \subset B$ such that $\dim_{\text{H}} B_{\theta_0} > 1$ and $B_{\theta_0} \angle b_{\theta_0}$. The rest of the argument is the same as before, applying [14, Theorem 2.8] to the projections ρ_θ , which are transversal restricted to the set B_{θ_0} . \square

Unfortunately, not all sets are BLP sets:

Remark 4.8. It is easy to construct a compact set $K \subset \mathbb{R}^3$ with $\dim_{\text{H}} K = 1$ such that

$$\dim_{\text{H}} \pi_V(K) = 0 \quad (4.9)$$

for a countable dense set of subspaces $V \in G(3, 2)$. Any such set K has the following property. Let $V_0 \in G(3, 2)$, and let K_0 be a subset of K such that the restriction $\pi_{V_0}|_{K_0}$ is bi-Lipschitz. Then $\dim_{\text{H}} K_0 = 0$. Indeed, if $\pi_{V_0}|_{K_0}$ is bi-Lipschitz, then $\pi_V|_{K_0}$ is also bi-Lipschitz for all 2-planes V in a small $G(3, 2)$ -neighbourhood of V_0 . This means that $\dim_{\text{H}} \pi_V(K) \geq \dim_{\text{H}} \pi_V(K_0) = \dim_{\text{H}} K_0$ for all 2-planes V in an open subset of $G(3, 2)$, and now (4.9) forces $\dim_{\text{H}} K_0 = 0$.

4.2. Self-similar sets. In this section, we prove that self-similar sets without rotations in \mathbb{R}^3 satisfy the assumptions of Theorem 4.5. We start by setting some notation. Consider a collection $\{\psi_1, \dots, \psi_q\}$ of contracting similitudes $\psi_i : \mathbb{R}^3 \rightarrow \mathbb{R}^3$. According to a result of Hutchinson [4] there exists a unique nonempty compact set $K \subset \mathbb{R}^3$ satisfying $K = \bigcup_{i=1}^q \psi_i(K)$. Such sets K are referred to as *self-similar sets*. If the generating similitudes of K have the form $\psi_i(x) = r_i x + w_i$ with $0 < r_i < 1$ and $w_i \in \mathbb{R}^3$, we call K a *self-similar set without rotations*. The fact that the mappings ψ_i do not involve rotations will be used to guarantee that the projection of K to an arbitrary plane is again self-similar.

Proposition 4.10. *Every self-similar set in \mathbb{R}^3 without rotations is a BLP set.*

Before presenting the proof, we recall some terminology from [12]. Rescaling the translation vectors w_i if necessary, we may assume that the similitudes ψ_i , $i \in \{1, \dots, q\}$, map the ball $B(0, \frac{1}{2})$ into itself. Then, we set $\mathcal{B}_0 = \{B(0, \frac{1}{2})\}$ and refer to the recursively defined family

$$\mathcal{B}_n := \{\psi_j(B) : B \in \mathcal{B}_{n-1}, 1 \leq j \leq q\}$$

as the collection of *generation n balls of K associated with $\{\psi_1, \dots, \psi_q\}$* .

The subset $K_{V,\varepsilon}$ to be constructed in the proof of Proposition 4.10 will be the attractor of a family of similitudes of the form $\{\psi_B : B \in \mathcal{G}\}$, where \mathcal{G} is a suitably chosen collection of balls in $\bigcup_{m \in \mathbb{N}} \mathcal{B}_m$. Here, ψ_B stands for a similitude of the form $\psi_B = \psi_{i_1} \circ \dots \circ \psi_{i_n}$, mapping $B(0, \frac{1}{2})$ to $B = \psi_{i_1} \circ \dots \circ \psi_{i_n}(B(0, \frac{1}{2}))$. For a given $B \in \mathcal{B}_n$, the selection of $\psi_{i_1}, \dots, \psi_{i_n}$ may not be unique, but then any choice is equally good for us. Observe that, for an arbitrary collection of balls $\mathcal{G} \subseteq \bigcup_{m \in \mathbb{N}} \mathcal{B}_m$, the associated attractor is a subset of K . Also, since \mathcal{B}_0 was defined to consist of a single ball of diameter one, ψ_B has contraction ratio $\text{diam}(B)$.

If $\{r_1, \dots, r_q\}$ are the contraction ratios of an IFS $\{\psi_1, \dots, \psi_q\}$, then the *similarity dimension* of the associated attractor K is defined as the unique number $s \geq 0$, which solves the equation

$$\sum_{j=1}^q r_j^s = 1.$$

It is well known, see [4], that $s = \dim_{\text{H}} K$, provided that K exhibits a sufficient degree of separation. One such condition is the *very strong separation condition*, which, by definition, requires the generation 1 balls of K to be disjoint. It is a stronger requirement than the *open set condition* commonly used in literature, but will be very convenient in the proof of Proposition 4.10.

Proof of Proposition 4.10. Let $V \in G(3, 2)$ and $\varepsilon > 0$ be arbitrary. The assumption that the similitudes ψ_1, \dots, ψ_q generating the self-similar set K contain no rotations ensures that the set $\pi_V(K)$ is again self-similar. It is a subset of V , or, under the customary identification, a subset of \mathbb{R}^2 , given by the IFS

$$\{\psi_{1,V}, \dots, \psi_{q,V}\} \quad \text{with } \psi_{j,V} : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad \psi_{j,V}(x) = r_j x + \pi_V(w_j).$$

The corresponding collection of generation n balls will be denoted by $\mathcal{B}_{n,V}$. Observe that the ball $B(0, \frac{1}{2})$ in \mathbb{R}^3 is projected to the ball $B(0, \frac{1}{2})$ in \mathbb{R}^2 , and hence $\mathcal{B}_{n,V}$ comprises precisely the projections of the balls in \mathcal{B}_n .

According to Lemma 3.4 in [12], we can for every $\varepsilon > 0$ choose a self-similar set $K^V \subset \pi_V(K)$ (depending on ε) with $\dim_{\text{H}} K^V \geq \dim_{\text{H}} \pi_V(K) - \varepsilon$ satisfying the very strong separation condition. In fact, the proof in [12] provides an IFS, which generates the set K^V and for which the generation 1 balls are a subcollection \mathcal{B}_1^V of disjoint balls in $\mathcal{B}_{n,V}$, for some large $n \in \mathbb{N}$. Moreover, we have

$$\sum_{B \in \mathcal{B}_1^V} \text{diam}(B)^s = 1 \tag{4.11}$$

with $s = \dim_{\text{H}} K^V$. Each ball $B \in \mathcal{B}_1^V$ is the image of a ball in \mathcal{B}_n under the projection π_V . There might be several such balls in \mathcal{B}_n , but we just pick one of them. We denote by \mathcal{G}_1 the collection of balls in \mathbb{R}^3 obtained in this way. Since the balls in \mathcal{B}_1^V are disjoint, the balls in \mathcal{G}_1 are contained in disjoint well-separated tubes perpendicular to V (and thus parallel to V^\perp). Also, (4.11) implies that

$$\sum_{B \in \mathcal{G}_1} \text{diam}(B)^s = 1 \tag{4.12}$$

with $s = \dim_{\text{H}} K^V$. The set $K_{V,\varepsilon} \subset K$, whose existence is claimed in the statement of the proposition, is obtained as the attractor of the IFS $\{\psi_B : B \in \mathcal{G}_1\}$. In other words, the balls in \mathcal{G}_1 form the generation 1 balls of $K_{V,\varepsilon}$. By (4.12) and the strong separation condition, we have

$$\dim_{\text{H}} K_{V,\varepsilon} = \dim_{\text{H}} K^V \geq \dim_{\text{H}} \pi_V(K) - \varepsilon.$$

It remains to be established that the restriction of π_V to $K_{V,\varepsilon}$ is bi-Lipschitz. To this end, we use the equivalent characterisation of this property in terms of cones as stated in Lemma 4.3. So far, we know that distinct balls in \mathcal{G}_1 are contained in disjoint closed tubes in direction V^\perp . This allows us to find $\alpha > 0$ so that

$$B \cap X(y, V^\perp, \alpha) = \emptyset \quad \text{for all } y \in B', \tag{4.13}$$

whenever B and B' are distinct balls in \mathcal{G}_1 . Then, it is a consequence of self-similarity that (4.13) holds **with the same constant** α for distinct generation n balls of $K_{V,\varepsilon}$, for any $n \in \mathbb{N}$. This is the content of the following lemma, a counterpart of which for sets in the plane is [12, Proposition 4.14]. Since the proof in higher dimensions is completely analogous, we omit it here.

Lemma 4.14. *Let $n \in \mathbb{N}$ be arbitrary and denote by \mathcal{G}_n the generation n balls of $K_{V,\varepsilon}$. Then, whenever B and B' are distinct balls in \mathcal{G}_n , we have*

$$B \cap X(y, V^\perp, \alpha) = \emptyset \quad \text{for all } y \in B'.$$

Consequently, the restriction of π_V to $K_{V,\varepsilon}$ is bi-Lipschitz with constant α , and the proof of the proposition is complete. \square

Finally, Theorem 1.8 follows by combining Theorem 4.5 with the BLP property of self-similar sets established in Proposition 4.10.

5. FURTHER RESULTS

5.1. Product sets and projections onto lines on a cone. Let $K = K_1 \times K_2$ be a product set in \mathbb{R}^3 with $K_1 \subset \mathbb{R}^2$ and $K_2 \subset \mathbb{R}$, and consider the curve $\gamma: (0, 2\pi) \rightarrow S(0, \sqrt{2})$ given by

$$\gamma(\theta) = (\cos(\theta), \sin(\theta), 1).$$

Then the lines $\ell_\theta := \text{span}(\gamma(\theta))$, $\theta \in (0, 2\pi)$, foliate the surface of a vertical cone in \mathbb{R}^3 , and the projections of K under $\rho_\theta(x) := \gamma(\theta) \cdot x$ have a particularly simple form:

$$\rho_\theta(K) = \rho_\theta(K_1 \times K_2) = \wp_\theta(K_1) + K_2, \quad (5.1)$$

where $\wp_\theta: \mathbb{R}^2 \rightarrow \mathbb{R}$ is the planar projection $\wp_\theta(x, y) = x \cos \theta + y \sin \theta$. It is easy to verify that the curve γ (normalised by a constant) satisfies the non-degeneracy condition (1.2), so Theorem 1.8 holds for the projections ρ_θ . Applying part (b) to the 3-fold product of an equicontractive self-similar set in \mathbb{R} (which is a self-similar set in \mathbb{R}^3) and recalling (5.1) yields Corollary 1.9.

As the first ‘further result’, we prove a variant of Theorem 1.6(a) for product sets $K = K_1 \times K_2$ and the special family of projections ρ_θ defined above.

Proposition 5.2. *Let $K = K_1 \times K_2 \subset \mathbb{R}^3$, where $K_1 \subset \mathbb{R}^2$, $K_2 \subset \mathbb{R}$ are analytic sets. Then $\dim_{\text{H}} \rho_\theta(K) \geq \min\{\frac{1}{2}, \dim_{\text{H}} K_1\} + \dim_{\text{H}} K_2$ for almost every $\theta \in (0, 2\pi)$.*

Proof. Let μ_1 and μ_2 be positive Borel measures supported on K_1 and K_2 , respectively, such that $I_{t_1}(\mu_1) < \infty$ for some $0 < t_1 < \min\{\dim_{\text{H}} K_1, 1/2\}$ and

$I_{t_2}(\mu_2) < \infty$ for some $0 < t_2 < \dim_{\mathbb{H}} K_2$. Then, with $\mu = \mu_1 \times \mu_2$, we have

$$\begin{aligned} \int_0^{2\pi} I_{t_1+t_2}(\rho_{\theta\#}\mu) d\theta &= \int_0^{2\pi} \left(\int |\widehat{\rho_{\theta\#}\mu}(r)|^2 |r|^{t_1+t_2-1} dr \right) d\theta \\ &\sim \int_0^{2\pi} \left(\int |\hat{\mu}(r\gamma(\theta))|^2 |r|^{t_1+t_2-1} dr \right) d\theta \\ &= \int |\hat{\mu}_2(r)|^2 \left(\int_0^{2\pi} |\hat{\mu}_1(r \cos \theta, r \sin \theta)|^2 d\theta \right) |r|^{t_1+t_2-1} dr. \end{aligned}$$

The inner integral is, by definition, the *spherical average* $\sigma(\mu_1)(|r|)$ of μ_1 and an estimate of P. Mattila, see [10], yields

$$\sigma(\mu_1)(|r|) \lesssim |r|^{-t_1} I_{t_1}(\mu_1).$$

Here we needed the assumption $t_1 < 1/2$, which guarantees that t_1 is within the range where the results from [10] apply. We may now conclude that

$$\int_0^{2\pi} I_{t_1+t_2}(\rho_{\theta\#}\mu) d\theta \lesssim I_{t_1}(\mu_1) \int |\hat{\mu}_2(r)|^2 |r|^{t_2-1} dr \sim I_{t_1}(\mu_1) I_{t_2}(\mu_2) < \infty,$$

and thus $I_{t_1+t_2}(\rho_{\theta\#}\mu) < \infty$ for almost every θ . This implies that

$$\dim_{\mathbb{H}} \rho_{\theta}(K_1 \times K_2) \geq t_1 + t_2$$

for almost every $\theta \in (0, 2\pi)$, and the proposition follows. \square

Before moving on to other topics, we remark that, in light of (5.1), the following conjecture is a weaker variant of Conjecture 1.5:

Conjecture 5.3. *Let $K_1 \subset \mathbb{R}^2$ and $K_2 \subset \mathbb{R}$ be analytic sets satisfying $\dim_{\mathbb{H}} K_1 + \dim_{\mathbb{H}} K_2 \leq 1$. Then*

$$\dim_{\mathbb{H}}(\varphi_{\theta}(K_1) + K_2) \geq \dim_{\mathbb{H}} K_1 + \dim_{\mathbb{H}} K_2$$

for almost every $\theta \in (0, 2\pi)$.

5.2. Another lower bound for general sets. In this section, we consider the general one-dimensional family of projections $(\rho_{\theta})_{\theta \in U}$.

Proposition 5.4. *If $K \subset \mathbb{R}^3$ is an analytic set with $0 \leq \dim_{\mathbb{H}} K \leq 2$, then $\dim_{\mathbb{P}} \rho_{\theta}(K) \geq \dim_{\mathbb{H}} K/2$ for almost every $\theta \in U$.*

The proposition starts improving on the lower bound for $\sigma_1(s) > 1/2$ from Remark 1.7 when $\dim_{\mathbb{H}} K = s \approx 1.077$.

Proof of Proposition 5.4. Write $\dim_{\mathbb{H}} K =: s$ and assume $0 < s \leq 2$. To reach a contradiction, suppose that there is a set $E \subset U$ with positive length such that $\dim_{\mathbb{P}} \rho_{\theta}(K) < s/2$ for every $\theta \in E$. Find two distinct Lebesgue points $\theta_1, \theta_2 \in E$ such that

$$(\gamma(\theta_1) \times \dot{\gamma}(\theta_1)) \cdot \dot{\gamma}(\theta_2) \neq 0.$$

Such points are given by the same argument as we used to obtain (3.21). Next, use continuity to find short open neighbourhoods $I, J \subset U$ of θ_1 and θ_2 such that

$$|(\gamma(\theta_I) \times \dot{\gamma}(\theta_I)) \cdot \dot{\gamma}(\theta_J)| \geq c > 0 \quad (5.5)$$

for all $(\theta_I, \theta_J) \in I \times J \subset \mathbb{R}^2$. Then, consider the two-parameter family of projections $\Pi_{(\theta_I, \theta_J)}: \mathbb{R}^3 \rightarrow \mathbb{R}^2$, $(\theta_I, \theta_J) \in I \times J$, given by

$$\Pi_{(\theta_I, \theta_J)}(x) := (\rho_{\theta_I}(x), \rho_{\theta_J}(x)) = (\gamma(\theta_I) \cdot x, \gamma(\theta_J) \cdot x).$$

Using (5.5), one may check that this is a family of generalised projections satisfying the framework of Peres and Schlag, see [14, Definitions 7.1 and 7.2]. Indeed, if $x \in \mathbb{R}^3$ is a unit vector such that, simultaneously, $\Pi_{(\theta_I, \theta_J)}(x) = 0$ and

$$0 = \det[D\Pi_{(\theta_I, \theta_J)}(x)(D\Pi_{(\theta_I, \theta_J)}(x))^T] = (\dot{\gamma}(\theta_I) \cdot x)^2 + (\dot{\gamma}(\theta_J) \cdot x)^2,$$

then x is perpendicular to both planes $\text{span}(\{\gamma(\theta_I), \gamma(\theta_J)\})$ and $\text{span}(\{\dot{\gamma}(\theta_I), \dot{\gamma}(\theta_J)\})$, which implies that

$$0 = \gamma(\theta_I) \cdot (\dot{\gamma}(\theta_I) \times \dot{\gamma}(\theta_J)) = (\gamma(\theta_I) \times \dot{\gamma}(\theta_I)) \cdot \dot{\gamma}(\theta_J),$$

violating (5.5).

Now [14, Theorem 7.3] implies that $\dim_{\text{H}} \Pi_{(\theta_I, \theta_J)}(K) = s$ for almost every pair $(\theta_I, \theta_J) \in I \times J$. On the other hand,

$$\Pi_{(\theta_I, \theta_J)}(K) \subset \rho_{\theta_I}(K) \times \rho_{\theta_J}(K),$$

so we obtain the estimate

$$\dim_{\text{p}} \rho_{\theta_I}(K) + \dim_{\text{p}} \rho_{\theta_J}(K) \geq \dim_{\text{H}} \Pi_{(\theta_I, \theta_J)}(K) = s$$

for almost every pair $(\theta_I, \theta_J) \in I \times J$. For such pairs (θ_I, θ_J) , we have either $\dim_{\text{p}} \rho_{\theta_I}(K) \geq s/2$ or $\dim_{\text{p}} \rho_{\theta_J}(K) \geq s/2$, which means that

$$\begin{aligned} |I||J| &= |I \times J| = |\{(\theta_I, \theta_J) : \dim_{\text{p}} \rho_{\theta_I}(K) \geq \frac{s}{2} \text{ or } \dim_{\text{p}} \rho_{\theta_J}(K) \geq \frac{s}{2}\}| \\ &\leq |\{\theta_I \in I : \dim_{\text{p}} \rho_{\theta_I}(K) \geq \frac{s}{2}\}| |J| + |I| |\{\theta_J \in J : \dim_{\text{p}} \rho_{\theta_J}(K) \geq \frac{s}{2}\}|. \end{aligned}$$

We may conclude that either

$$|\{\theta_I \in I : \dim_{\text{p}} \rho_{\theta_I}(K) \geq \frac{s}{2}\}| \geq \frac{|I|}{2} \quad \text{or} \quad |\{\theta_J \in J : \dim_{\text{p}} \rho_{\theta_J}(K) \geq \frac{s}{2}\}| \geq \frac{|J|}{2}.$$

However, since θ_1 and θ_2 were Lebesgue points of E , neither option is possible if I and J were chosen short enough to begin with. This contradiction completes the proof. \square

APPENDIX A. A DISCRETE VERSION OF FROSTMAN'S LEMMA

In this section, we prove Lemma 3.13. Let us recall the statement:

Proposition A.1. *Let $\delta > 0$, and let $B \subset \mathbb{R}^3$ be a set with $\mathcal{H}_{\infty}^s(B) =: \kappa > 0$. Then, there exists a (δ, s) -set $P \subset B$ with cardinality $|P| \gtrsim \kappa \cdot \delta^{-s}$.*

Proof. Without loss of generality, assume that $\delta = 2^{-k}$ for some $k \in \mathbb{N}$ and $B \subset [0, 1]^3$. Denote by \mathcal{D}_k the dyadic cubes in \mathbb{R}^3 of side-length 2^{-k} . First, find all the dyadic cubes $Q^k \in \mathcal{D}_k$, which intersect B , and choose a single point $x \in B \cap Q^k$ for each Q^k . The finite set so obtained is denoted by P_0 . Next, modify P_0 as follows. Consider the cubes in \mathcal{D}_{k-1} . If one of these, say Q^{k-1} , satisfies

$$|P_0 \cap Q^{k-1}| > \left(\frac{d(Q^{k-1})}{\delta} \right)^s,$$

remove points from $P_0 \cap Q^{k-1}$, until the reduced set P'_0 satisfies

$$\frac{1}{2} \left(\frac{d(Q^{k-1})}{\delta} \right)^s \leq |P'_0 \cap Q^{k-1}| \leq \left(\frac{d(Q^{k-1})}{\delta} \right)^s.$$

Repeat this for all cubes $Q^{k-1} \in \mathcal{D}_{k-1}$ to obtain P_1 . Then, repeat the procedure at all dyadic scales up from δ , one scale at a time: whenever P_j has been defined, and there is a cube $Q^{k-j-1} \in \mathcal{D}_{k-j-1}$ such that

$$|P_j \cap Q^{k-j-1}| > \left(\frac{d(Q^{k-j-1})}{\delta} \right)^s,$$

remove points from $P_j \cap Q^{k-j-1}$, until the reduced set P'_j satisfies

$$\frac{1}{2} \left(\frac{d(Q^{k-j-1})}{\delta} \right)^s \leq |P'_j \cap Q^{k-j-1}| \leq \left(\frac{d(Q^{k-j-1})}{\delta} \right)^s. \quad (\text{A.2})$$

Stop the process, when the remaining set of points, denoted by P , is entirely contained in some dyadic cube $Q_0 \subset [0, 1]^3$. Now, we claim that for every point $x \in P_0$ there exists a unique maximal dyadic cube $Q_x \subset Q_0$ such that $\ell(Q_x) \geq \delta$ and

$$|P \cap Q_x| \geq \frac{1}{2} \left(\frac{d(Q_x)}{\delta} \right)^s. \quad (\text{A.3})$$

We only need to show that there exists **at least one** cube $Q_x \ni x$ satisfying (A.3); the rest follows automatically from the dyadic structure. If $x \in P$, we have (A.3) for the dyadic cube $Q_x \in \mathcal{D}_k$ containing x . On the other hand, if $x \in P_0 \setminus P$, the point x was deleted from P_0 at some stage. Then, it makes sense to define Q_x as the dyadic cube containing x , where the ‘last deletion of points’ occurred. If this happened while defining P_{j+1} , we have (A.2) with $Q_{k-j-1} = Q_x$. But since this was the last cube containing x , where **any** deletion of points occurred, we see that that $P'_j \cap Q_x = P \cap Q_x$. This gives (A.3).

Now, observe that the cubes $\{Q_x : x \in P_0\}$,

- cover B , because they cover every cube in \mathcal{D}_k containing a point in P_0 , and these cubes cover B ,
- are disjoint, hence partition the set P .

These facts and (A.3) yield the lower bound

$$|P| = \sum |P \cap Q_x| \gtrsim \delta^{-s} \sum d(Q_x)^s \geq \kappa \cdot \delta^{-s}.$$

It remains to prove that P is a (δ, s) -set. For dyadic cubes $Q \in \mathcal{D}_l$ with $l \leq k$ it follows immediately from the construction of P , in particular the right hand side of (A.2), that

$$|P \cap Q| \leq \left(\frac{d(Q)}{\delta} \right)^s.$$

The statement for balls $B \subset \mathbb{R}^3$ with $d(B) \geq \delta$ follows by observing that any such ball can be covered by ~ 1 dyadic cubes of diameter $\sim d(B)$. \square

APPENDIX B. AUXILIARY RESULTS FOR CURVES

In this section, we prove Lemma 3.4, Lemma 3.17 and Lemma 3.22.

Proof of Lemma 3.4. Consider the function

$$\Pi : [0, 1] \times S^2 \rightarrow \mathbb{R}, \quad \Pi(\theta, x) := \rho_\theta(x) = \gamma(\theta) \cdot x,$$

and let $\delta > 0$ be a constant such that

$$\max \{ |\Pi(\theta, x)|, |\partial_\theta \Pi(\theta, x)|, |\partial_\theta^2 \Pi(\theta, x)| \} \geq \delta, \quad (\theta, x) \in [0, 1] \times S^2. \quad (\text{B.1})$$

Then, find $\varepsilon > 0$ so that for all $(\theta, x), (\theta', x) \in [0, 1] \times S^2$ with $|\theta - \theta'| < \varepsilon$, we have

$$\max \{ |\Pi(\theta, x) - \Pi(\theta', x)|, |\partial_\theta \Pi(\theta, x) - \partial_\theta \Pi(\theta', x)|, |\partial_\theta^2 \Pi(\theta, x) - \partial_\theta^2 \Pi(\theta', x)| \} < \delta. \quad (\text{B.2})$$

We claim that the statement of the lemma holds for this choice of ε . Fix $x \in S^2$ and let $I \subset [0, 1]$ be an interval of length ε . To reach a contradiction, assume that there exist distinct points $\theta_1, \theta_2, \theta_3 \in I$ such that

$$\Pi(\theta_1, x) = \Pi(\theta_2, x) = \Pi(\theta_3, x) = 0.$$

Applying Rolle's theorem to the function $\theta \mapsto \Pi(\theta, x)$, we conclude that there are at least two points in I where also the derivative $\partial_\theta \Pi(\cdot, x)$ vanishes, and, by another application of Rolle's theorem, we find a point in I where also $\partial_\theta^2 \Pi(\cdot, x)$ is zero. From (B.2) it follows that

$$\max \{ |\Pi(\theta, x)|, |\partial_\theta \Pi(\theta, x)|, |\partial_\theta^2 \Pi(\theta, x)| \} < \delta$$

for all $\theta \in I$, which contradicts (B.1). \square

Proof of Lemma 3.17. Our goal is to find $\varepsilon_1, \varepsilon_2 > 0$ and $L < 1$ such that

$$|\rho_\theta(x - y)| \leq L|x - y| \quad \text{for all } x, y \in C_I \text{ and } \theta \in C_J, \quad (\text{B.3})$$

where $I = [\theta_1 - \varepsilon_1, \theta_1 + \varepsilon_1]$ and $J = [\theta_2 - \varepsilon_2, \theta_2 + \varepsilon_2]$. Elements in C_I are of the form $x = r_x \gamma(\theta_x)$ with $r_x \in \mathbb{R}$ and $\theta_x \in I$. As we will explain now, it is enough to verify (B.3) for pairs $x = r_x \gamma(\theta_x) \in C_I$ and $y = r_y \gamma(\theta_y) \in C_I$ with $r_x, r_y \geq 0$. Clearly, if (B.3) holds for all such pairs x, y then it also holds for pairs x, y with $r_x, r_y \leq 0$. In case r_x and r_y have opposite signs, (B.3) will be valid with some constants $L' \in [L, 1)$ and $\varepsilon'_1 < \varepsilon_1$. The precise condition on $\varepsilon'_1 > 0$ is that

$$L^2 < \min_{\theta_x, \theta_y \in I'} \gamma(\theta_x) \cdot \gamma(\theta_y) \quad \text{with} \quad I' := [\theta_1 - \varepsilon'_1, \theta_1 + \varepsilon'_1]. \quad (\text{B.4})$$

This can be achieved by the continuity of γ , since $\gamma(\theta_1) \cdot \gamma(\theta_1) = 1$ and $L < 1$. Now, fix $x = r_x \gamma(\theta_x) \in C_I$ and $y = r_y \gamma(\theta_y) \in C_I$ with $r_x r_y \leq 0$. Then, assuming (B.3) for points on C_I with the same sign, we have

$$|\rho_\theta(x - y)| \leq |\rho_\theta(x)| + |\rho_\theta(y)| \leq L(|x| + |y|) \leq dL|x - y| = L'|x - y|,$$

where

$$d = \left(\min_{\theta_x, \theta_y \in I'} \gamma(\theta_x) \cdot \gamma(\theta_y) \right)^{-1/2} \geq 1 \quad \text{and} \quad L' = dL < 1$$

by (B.4). The inequality $|x| + |y| \leq d|x - y|$ follows from

$$(|x| + |y|)^2 = r_x^2 + r_y^2 - 2r_x r_y \leq d^2(r_x^2 + r_y^2 - 2r_x r_y (\gamma(\theta_x) \cdot \gamma(\theta_y))) = d^2|x - y|^2.$$

It remains to prove (B.3) for points $x = r_x \gamma(\theta_x)$ and $y = r_y \gamma(\theta_y)$ with $r_x, r_y \geq 0$ and $\theta_x, \theta_y \in I$. Without loss of generality we assume that $r_x \leq r_y$. The differentiability of γ at θ_y yields

$$\begin{aligned} |\rho_\theta(x - y)| &= |[r_x \gamma(\theta_x) - r_y \gamma(\theta_y)] \cdot \gamma(\theta)| \\ &\leq |[r_x \dot{\gamma}(\theta_y)(\theta_x - \theta_y) + (r_x - r_y) \gamma(\theta_y)] \cdot \gamma(\theta)| + r_x o(|\theta_x - \theta_y|). \end{aligned}$$

Exploiting the assumption $\gamma(\theta_2) \notin \text{span}(\{\gamma(\theta_1), \dot{\gamma}(\theta_1)\})$, we can find constants $L_0 < 1$ and $\varepsilon_1, \varepsilon_2 > 0$ such that

$$|[r_x \dot{\gamma}(\theta_y)(\theta_x - \theta_y) + (r_x - r_y) \gamma(\theta_y)] \cdot \gamma(\theta)| \leq L_0 \sqrt{r_x^2 |\dot{\gamma}(\theta_y)|^2 (\theta_x - \theta_y)^2 + (r_x - r_y)^2}$$

for all $\theta_x, \theta_y \in [\theta_1 - \varepsilon_1, \theta_1 + \varepsilon_1] =: I$ and $\theta \in [\theta_2 - \varepsilon_2, \theta_2 + \varepsilon_2] =: J$. Given $\varepsilon > 0$, we can make ε_1 smaller still to ensure that the inequality

$$|\dot{\gamma}(\theta_y)| |\theta_x - \theta_y| \leq (1 + \varepsilon) |\gamma(\theta_x) - \gamma(\theta_y)|$$

holds whenever $|\theta_x - \theta_y| \leq 2\varepsilon_1$. Choosing $\varepsilon > 0$ small enough, inserting this estimate to the upper bound for $|\rho_\theta(x - y)|$, and comparing the result with

$$|x - y| = \sqrt{r_x^2 + r_y^2 - 2r_x r_y \gamma(\theta_x) \cdot \gamma(\theta_y)} = \sqrt{r_x r_y |\gamma(\theta_x) - \gamma(\theta_y)|^2 + (r_x - r_y)^2},$$

we see that $|\rho_\theta(x - y)| \leq L|x - y|$ for some $L \in (L_0, 1)$. We also need to know that the bounds implicit in $o(|\theta_x - \theta_y|)$ can be chosen small in a manner depending only on ε_1 , but this follows from the C^2 regularity of γ . \square

Proof of Lemma 3.22. In order to establish (3.23) for all $\theta \in U$, it is sufficient to show

$$\ddot{\eta}(\theta) \cdot (\eta(\theta) \times \dot{\eta}(\theta)) \neq 0, \quad \text{for all } \theta \in U. \quad (\text{B.5})$$

This condition means precisely that $\eta(\theta), \dot{\eta}(\theta)$ and $\ddot{\eta}(\theta)$ are all of positive length, $\eta(\theta)$ and $\dot{\eta}(\theta)$ are not parallel and hence span a plane, and this plane does not contain $\ddot{\eta}(\theta)$. In order to prove (B.5), we first evaluate

$$\eta = \frac{\gamma \times \dot{\gamma}}{|\gamma \times \dot{\gamma}|} = \frac{1}{|\dot{\gamma}|} \gamma \times \dot{\gamma}, \quad \dot{\eta} = \left(\frac{1}{|\dot{\gamma}|} \right)' \gamma \times \dot{\gamma} + \frac{1}{|\dot{\gamma}|} \gamma \times \ddot{\gamma}$$

and

$$\ddot{\eta} = \left(\frac{1}{|\dot{\gamma}|} \right)'' \gamma \times \dot{\gamma} + 2 \left(\frac{1}{|\dot{\gamma}|} \right)' \gamma \times \ddot{\gamma} + \frac{1}{|\dot{\gamma}|} \dot{\gamma} \times \ddot{\gamma} + \frac{1}{|\dot{\gamma}|} \gamma \times \ddot{\ddot{\gamma}}.$$

Then,

$$\eta \times \dot{\eta} = \frac{1}{|\dot{\gamma}|^2} (\gamma \times \dot{\gamma}) \times (\gamma \times \ddot{\gamma}) = \frac{1}{|\dot{\gamma}|^2} (\gamma \cdot (\dot{\gamma} \times \ddot{\gamma})) \gamma.$$

Finally,

$$\ddot{\eta} \cdot (\eta \times \dot{\eta}) = \frac{1}{|\dot{\gamma}|^3} (\gamma \cdot (\dot{\gamma} \times \ddot{\gamma}))^2,$$

which is non-vanishing, due to condition (1.2) for the curve γ . This concludes the proof of the lemma. \square

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